Suspension Bridges Made From Paper Clips

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With paper clips can be done even hands-on experiments with an ambitious background. In this paper is shown the difference between a catenary curve and a parabola with paper clips. The application to suspension bridges is pointed out. Physics simulation programs can verify the experiments. Short mathematical reflections complete the paper.

Chains made from paper clips

What shape does the curve of a chain or a flexible cable or rope suspended from its two ends take? Galileo Galilei asked himself this question – and answered it incorrectly. He thought it was a parabola. The correct curve wasn't derived until the end of the 17th century, by the brothers Jacob and Johann Bernoulli as well as by Gottfried Wilhelm Leibniz and Christiaan Huyghens. It is the so-called catenary curve, which is the hyperbolic cosine function (cosh), which can also be expressed as a sum of two exponential functions. The derivation of the catenary curve can be found in many mathematical as well as mechanics textbooks and is therefore not shown here.

The catenary curve can be constructed well with an adequate number of paper clips. The more paper clips used, the better the approximation to the ideal curve. In figure 1, a chain made from 16 paper clips is shown. An ideal catenary curve fits almost perfectly to this paper clip chain. The difference between a catenary curve and a parabola is especially distinctive with a relatively strong sag as in figure 1 (see also Fig. 4).

Fig. 1: Catenary curve with 16 paper clips

If heavy weights hang from each link of a chain, as for example with suspension bridges, the curve really changes from a catenary curve to a parabola (see info box). This case can also be constructed with paper clips.

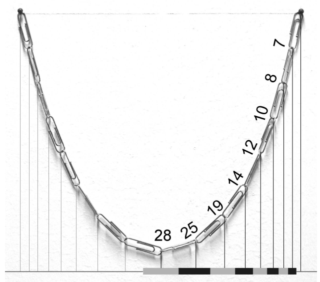

Fig. 2: Construction of a suspension bridge with paper clips. The numbers describe the horizontal distance between the suspension cables in millimeters.

Admittedly, the construction described below is mathematically not entirely correct, but in real life it leads to a very good result at the first attempt. Hang the chain of paper clips in front of a piece of paper, mark the joints on the piece of paper and draw straight vertical lines from the marks. In fig. 2, this is shown together with the paper clips. A horizontal line depicts the load (=road) held by the main cable of the suspension bridge. The part of the road (grey) suspended from the lowest point is both the biggest and the heaviest part. It stretches from the centre of the right paper clip to the centre of the left paper clip. From this lowest point, hang, e.g., as many paper clips as this part of the road has millimetres on the piece of paper. The adjacent part of the road (black) is shorter. From its joint also hang a number of paper clips corresponding to the number of millimetres which this part of the road has on the paper. In this way, each joint bears its corresponding weight. In fig. 2, the numbers next to the chain refer to the number of millimetres (and paper clips) obtained.

Only a few paper clips hang near the end points of the chain. The ratio of the weight of paper clips simulating the road to the weight of the main cable is only 1 to 3 here. At the lowest point, this ratio is about 20. This difference in ratios is caused by our construction, which presupposes equal distances between the joints from which the vertical cables hang. With real suspension bridges, the horizontal distances of the vertical cables are the same. The – constant – weight ratio is between about 10 to 1 and 15 to 1. Since there is no ideal weightless main cable, the shape of a real main cable is a mixture of a catenary and a parabola. In reality, this does not pose a problem for the design engineers, because a small sag makes the difference between a catenary and a parabola negligible.

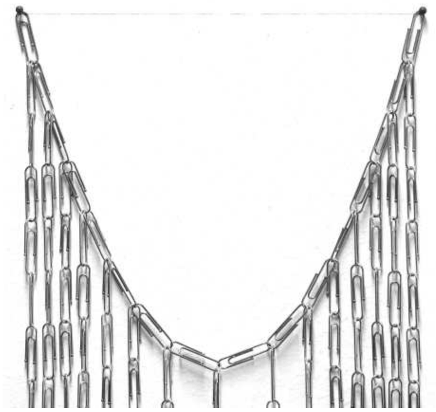

Fig. 3: Parabola with 16 clips and weights. The complete weights with the clip chains are not shown here.

The shape of the curve in figure 3 is that of an ideal parabola to within a very small margin of error.

With physics simulation programs like Interactive Physics [1] or XYZet [2] you can also very nicely illustrate the situations described above. In figure 4, you can see the simulation of a catenary curve with 16 unweighted links (grey line) and a 'suspension bridge' equipped with corresponding weights (black parabola), both superimposed on one another. With a real suspension bridge, the vertical suspender cables are equidistant from one another (this can also be simulated with the program). In figure 4, however, the points on the main suspension cable where the vertical suspender cables are fastened are equidistant from one another.

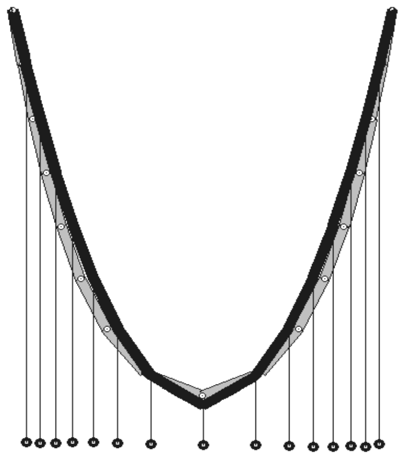

Fig. 4: Simulation of a catenary curve (grey) and a parabola (black) using ‘Interactive Physics’

In the WEB you can find many links under the term ‘catenary curve’, and also historical remarks and derivations. Furthermore, there are very descriptive applets that clarify the difference between the catenary curve and the parabola.

Info-box (suspension bridge parabola)

Overly simplified and idealized, the form of the curve of the main cable of a suspension bridge can be derived in the following way:
Three forces, whose vectorial sum must yield exactly zero (figure 5), act in a point P of the main cable of a suspension bridge. First, the force G of a part of the road with the length x acts vertically downward. Secondly, a horizontal force S is exerted by the tension of the cable. This force is constant over the whole cable. Thirdly, a force F acts in the direction of the tangent to the cable. This tangential force corresponds exactly to the slope at point P.

Let's take μ as the weight per unit of length of the roadway suspended at the cable. The coordinate origin 0 is located at the vertex of the curve. The weight G acting at the point P is then exactly G = μ·x. If we denote the height of the cable at the point x by y, then the slope at this point is

$$y' = \frac{G}{S} = \frac{\mu \cdot x}{S}$$

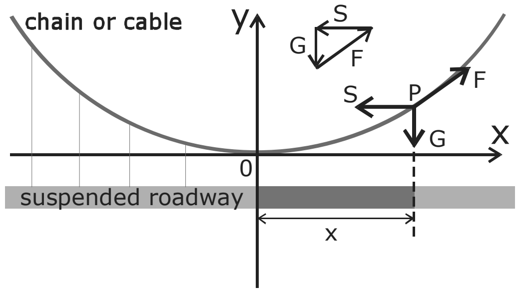

Fig. 5: A parabola emerges as a curve form for the main cable of a suspension bridge.

By integrating this equation we get

$$y = \int \frac{\mu \cdot x}{S} dx = \frac{\mu}{2S} x^2 + C$$

Since the origin 0 of the curve is located at the vertex, the integration constant is C = 0 . Consequently, the curve is a parabola.

References

[1] http://www.interactivephysics.com/ (in English)

[2] http://www.ipn.uni-kiel.de/persons/michael/xyzet/ (in English)