MECHANICAL WAVE ANALOGY TO THE QUANTUM WAVE FUNCTION

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1 INTRODUCTION

Quantum mechanics has proven to be a difficult subject for students. Mainly because many features of quantum mechanics cannot be observed in classical physics at all. Moreover, the quantum world is governed by probability rather than certainty. Classic analogies help facilitate the transition. They represent a tangible visualisation of the abstract concepts of quantum mechanics, in this case namely the wave function.

The quantum wave function itself is complex and does not represent any measurable physical quantity, whereas classical waves are real and their amplitude can be measured. This points out that analogies are not meant to be completely equivalent to the original phenomena, but rather a help to visualise what we otherwise can not observe.

In this paper we study one such analogy: the transition of a wave through a periodic set of barriers called a periodic potential. This classical phenomenon is mathematically identical to the transition of a quantum particle, described by a wave function, through a periodic set of potentials, like atoms in a crystal. The mathematical similarity offers a way to better imagine the quantum phenomenon, but also opens the possibility to study the classical phenomenon itself.

2 THE PHENOMENON

A periodic potential is any form of potential, that repeats periodically. To keep things as simple as possible, we choose the single potentials within the periodic set to be of the form of a delta function. Let us call them simply barriers.

As a wave hits a periodic potential, some of it is transmitted, and the rest is reflected. The transmission coefficient, defined as the ratio between the energy flux of the transmitted wave and the total energy flux of the incoming wave, depends on the frequency of the wave (Fig. 3). The frequency dependence consists of wide ranges of transmission coefficient close to one, separated by wide ranges of transmission coefficient close to zero, which are called band gaps. The phenomenon can be observed for classical waves, as well as for quantum particles described by the wave function.

2.1 The experiment

We performed a simple experiment using a wave machine. A wave machine consists of parallel rods placed perpendicularly on a wire and stiffly connected to

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it. A circular motion of a rod causes torsional tension in the wire which transfers the disturbance to the next rod [4].

The wave machine is driven on one end by a loudspeaker connected to a sine generator with variable frequency. On the other end we place an attenuator with a characteristic impedance (a piston in a glass of water) to prevent reflections from that end. We create the potential by adding weights to some of the rods of the wave machine at equal distances. The weights should be chosen so that the value of the coefficient $\alpha = -1/2\, I_u/\iota$, where I_u is the moment of inertia of the weight alone around the wire connecting the rods, and ι is the linear density of the moment of inertia of the whole wave machine around the same wire, is in the range of millimetres. In our case, with the weights placed at the end of the rods, this can be written as $\alpha = -3\, m_u/\mu$, where m_u is the mass of one weight and μ is the linear density of the wave machine. We used $\alpha = -3.1$ mm.

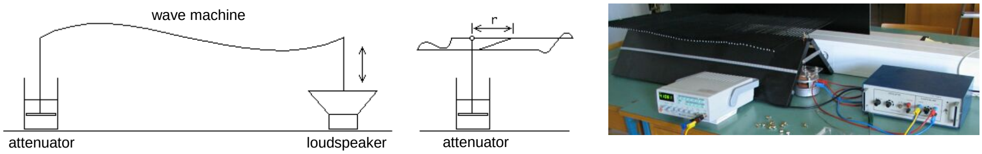

Figure 1 A schematics of the experimental setup (left) and a photo of the actual setup (right). The attenuator is not visible, because it is on the other side of the wave machine.

Just as the theory predicts, when we send waves through the potential, we find that for some frequencies waves are passing, while for others they are entirely reflected. We compared our measurements to the theoretical predictions and the formation of the gap at the predicted frequencies was clearly visible (Fig. 4). Moreover, the effect can be easily observed without measurement [5].

2.2 Graphical explanation

We can describe the phenomenon in terms of interference of waves reflected and transmitted from different pairs of barriers [2]. On each barrier part of the wave can be transmitted and part reflected. The reflected wave travels back, and part of it is again transmitted and part reflected from the previous barrier. In this way we get many partial waves travelling in both direction from each barrier.

Figure 2 A graphical representation of interference on a pair of barriers. To get the transmission coefficient, this interference must be calculated for all pairs of barriers [2].

These waves interfere. How they interfere depends on the frequency of the wave and the distance between the barriers. If this interference is destructive, then we get a minimum of transmitted energy. We can check the interference for every pair of barriers and we find out that minima for different pairs occur at the same frequencies. This is where the band gaps form.

3 MATHEMATICAL DESCRIPTION

Both the classical and quantum phenomenon can be described mathematically using exactly the same equations [1, 3]. But for classical waves, the calculations can be also done using trigonometric functions, without recurring to complex notation, which is necessary in the quantum case.

In this paper we will only focus on the transition through one barrier, since this determines almost all the properties for the whole periodic potential [3], and the comparison between classical and quantum results.

3.1 Laws of motion

Quantum mechanics (QM) is governed by the Schroedinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t)+V(x)\Psi(x,t)=i\hbar\frac{\partial}{\partial t}\Psi(x,t)\ , \tag{1}$$

where $V(x)$ is the potential, $\Psi(x,t)$ is the wave fnction, describing the observed particle, m is the mass of the particle, and $\hbar$ is the Planck constant, divided by 2π. Mechanical waves (MW) are governed by Newton's second law. For this discussion we choose waves on a string. In our case the potential is provided by the change in the density of the propagation medium (e.g. additional weights on the string)[3]

$$F\frac{\partial^2}{\partial x^2}y(x,t)=\mu(x)\frac{\partial^2}{\partial t^2}y(x,t)\ , \tag{2}$$

where y is the transversal displacement of the medium at position x in time t, F is the tension force that stretches the string, and μ is the linear density of the string. This form, the wave equation, although looking a little different from the one in QM, gives very similar and comparable results.

The equations are not entirely analogous. But it is interesting that all the differences together provide for the fact that the complex solutions to both equations are exactly the same. From the comparison between the equations we can see that the resultant mechanical force, needed to move the medium, $F\,\partial^2 y(x,t)/\partial x^2$ is analogous to the kinetic energy of a particle $-(\hbar^2/2m)\,\partial^2\Psi(x,t)/\partial x^2$ in QM, and the linear mass density $\mu(x)$, which incorporates both the medium and additional masses, is roughly analogous to the QM potential $V(x)$.

The next thing to determine is the MW equivalent of a QM free particle. In QM we can calculate the probability density for the position of a particle by calculating

$$\rho(x)=\psi^*(x)\psi(x)\ . \tag{3}$$

Doing the same for MW would give us the energy density of the wave. Hence probability in QM is analogous to energy in MW.

What is there in QM where the probability to find a particle is zero? In that case we have vacuum at those coordinates. How about MW? If there is no energy then the propagation medium (the string) is undisturbed. So vacuum in QM is analogous to undisturbed medium in MW. Hence particle, meaning matter in QM is analogous to disturbance – additional energy – of a medium in MW.

The complex solutions to both laws of motion in the region with no barriers are

$$\Psi(x,t) = Ae^{i(kx-\omega t)} + Be^{i(-kx-\omega t)} \ . \tag{4}$$

For the mechanical case we can choose to use only the real part

$$\begin{aligned} y(x,t) = \mathrm{Re}(\Psi(x,t)) = A_c \cos(kx-\omega t) + A_s \sin(kx-\omega t) + \\ + B_c \cos(-kx-\omega t) + B_s \sin(-kx-\omega t) \end{aligned} \ , \tag{5}$$

where $A = A_c - iA_s$, and $B = B_c - iB_s$. This is, of course, the real solution to the wave equation.

3.2 Transition through one potential

By calculating the transition through one potential, we can derive an important parameter α that determines the dependence of the transmission coefficient on the frequency of the waves. Although we will skip the calculation for the final result, we can briefly show, how we derive α for mechanical waves.

The solutions to the law of motion on both sides of one barrier take the form of Eq. (5), but with different amplitudes. For the amplitudes we can write a vector equation

$$\begin{bmatrix} A_c \\ A_s \\ B_c \\ B_s \end{bmatrix} = \begin{bmatrix} 1 & -\alpha\,\omega/c & 0 & -\alpha\,\omega/c \\ \alpha\,\omega/c & 1 & \alpha\,\omega/c & 0 \\ 0 & \alpha\,\omega/c & 1 & \alpha\,\omega/c \\ -\alpha\,\omega/c & 0 & -\alpha\,\omega/c & 1 \end{bmatrix} \begin{bmatrix} C_c \\ C_s \\ D_c \\ D_s \end{bmatrix} , \tag{6}$$

where A_c, A_s, B_c, and B_s are amplitudes on the left side of the barrier (as defined in Eq. (5)), and C_c, C_s, D_c, and D_s are amplitudes on the right side of it. The matrix is called the transfer matrix and the form in Eq. (6) is the form it takes for a delta potential

$$V(x) = m_u \delta(x) \ .$$

The coefficient

$$\alpha = -\frac{1}{2}\frac{m_u}{\mu} \ , \tag{7}$$

with m_u being the mass of one weight, and μ the linear density of the string, determines the width of the gaps for the transmission coefficient.

We will skip all the steps between here and the final result for the whole periodic potential [3], and simply jump right to the end.

4 TRANSMISSION COEFFICIENT

The transmission coefficient is defined as

$$T = \frac{{C_c}^2 + {C_s}^2}{{A_c}^2 + {A_s}^2} \ , \tag{8}$$

where the amplitudes now refer to those on the left and right side of the whole periodic potential (not just one barrier). From this we get the following expression [3]

$$T(\omega) = \left(1 + \left(\alpha \tfrac{1}{c}\omega \frac{\sin\left(N \arccos\left(\cos(\frac{s}{c}\omega) + \alpha \frac{1}{c}\omega \sin(\frac{s}{c}\omega)\right)\right)}{\sin\left(\arccos\left(\cos(\frac{s}{c}\omega) + \alpha \frac{1}{c}\omega \sin(\frac{s}{c}\omega)\right)\right)}\right)^2\right)^{-1} . \tag{9}$$

We plot the value of T in relation to $\nu = \omega/2\pi$ on Fig. 3 to better see the "shape" of the transmission coefficient.

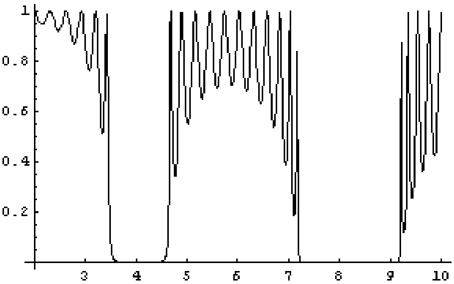

Figure 3 A plot of the transmission coefficient (vertical axis) with respect to the frequency of the wave in Hz (horizontal axis). The plot serves only as an illustration of the phenomenon, and was derived using Eq. (9) for values $\alpha = -8$ mm, $N = 12$, $s = 36.12$ mm.

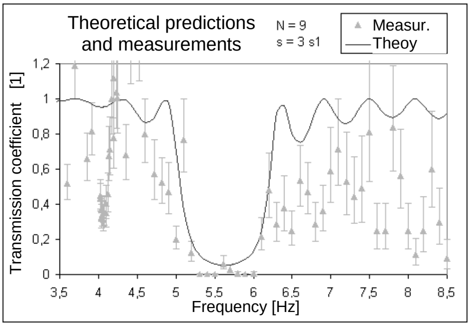

Figure 4 Comparison of measured values and theoretical prediction for the transmission coefficient for $N = 9$, $s = 36.12$ mm, and $\alpha = -3.1$ mm.

There are three parameters that determine the details of the shape of $T(\omega)$. To see each of their effects, we would have to plot a few graphs for different values of each of them. It turns out that N, the number of barriers, determines the depth and sharpness of the gaps, since the more barriers we have, the more pairs we have and the more we amplify the effect of each pair. We can also see that the gaps become quite obvious already at as few as 8 barriers. Parameter s is the distance between the cells, which, of course, determines at what frequencies the gaps will form (see subsection 2.2). Parameter α determines the width of the gaps.

4.1 Parameter α

Parameter α is determined by the shape of the single potentials in the periodic potential. For a delta function we get for mechanical waves Eq. (7) and for quantum mechanics, where we write a delta potential as $V(x) = V_0\delta(x)$ we get

$$\alpha = \frac{1}{2}\frac{V_0}{W} , \tag{10}$$

where W is the total energy of a particle.

In MW, the parameter is determined by the ratio between the inertia of the potential (weight of the weights) and the inertia (mass) density of the medium. So the parameter measures how much more difficult it is to move the medium at the point of the potential with respect to the medium alone. In the QM analogy the parameter measures the ratio between the energy needed to overcome the potential and the whole energy of the particle described by the wave function.

A large α means a high potential. By sending α towards infinity, we approach the situation where, for MW, the weights are so heavy that they are virtually impossible to move, and the behaviour approaches that of reflection on a fixed end. At large α, but not close to infinite, we get a situation, where gaps are so wide, we only have very narrow bands of transmission at frequencies corresponding to the characteristic frequencies of the weights on the string.

On the other hand, when α approaches zero, the situation does not approach that at an open end, since the medium is already given, but rather that of no additional potentials. Hence in this situation, the transmission coefficient approaches a constant value of 1, as if there were no potentials at all.

5 CONCLUSION

Since the mathematical description of the behaviour of a particle in QM and a wave in classical waves are identical, many of the effects seen in QM can be also observed in classical waves. For this to be of any help to facilitate the transition of thought towards QM, we must know exactly what quantities are analogous between QM and classical waves and what are the limitations of such analogies. The transition through a locally periodic potential is one such analogy, that can be easily observed for classical waves and is important for frequency filters in all types of physical elements described by wave functions: mechanical waves [1], sound [1, 2], electromagnetic waves [1] and quantum particles [1].

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[3] For a complete calculation in complex notation in general and for different specific potentials for both quantum mechanics and classical waves, please refer to [1].

[4] K. D. Skeldon, J. E. Mline, A. I. Grant, D. A. Palmer 1998 *Making Waves: A Classroom Torsional Wave Machine*, The Physics Teacher **36** 392 – 398 and 466 – 472

[5] A movie of the experiment with the wave machine can be obtained upon request by sending an e-mail to the corresponding author.