

Measuring and Improving Student Mathematical Skills for Modeling

Ronald K. Thornton

(csmt@tufts.edu), Tufts University, Medford, MA, USA

Abstract

The primary focus of this paper is on necessary pre- or co-requisite student knowledge needed to understand modeling and use it to solve problems. We discuss how student understanding of two such areas of knowledge, recognizing equations as functional relationships and vectors, can be evaluated using the *Mathematical Modeling Conceptual Evaluation (MMCE)*. Methods to improve student learning in these areas are described (analytic modeling and *Visualizer*®) and preliminary student-learning research results for traditional and reform courses are presented.

Introduction

We have worked for years to produce activity-based curricular materials, methods, and technological tools (e.g. real-time data logging or “MBL” tools) that help students to understand physics concepts by interacting with the physical world and we have worked to understand student thinking. In fact, we have had some success in helping students understand concepts and in understanding their thinking (Thornton, 1999).

We are now involved in an even more difficult task which is to help students to understand physics in terms of modeling (mathematical and other). We would like to set a more general context for the specific topic discussed in this paper. We planned a number of tasks for our modeling work.

- Present physics as the exploration of the physical world & the construction, validation, and application of conceptual models
- Create tool-based software
- Develop tools and techniques to aid students to learn and to relate different depictions and representations
- Create activities that begin with and return often to the physical world
- Evaluate students to determine necessary pre- or co-requisite knowledge.

We have also imposed a number of pedagogical requirements for our modeling software. It must enhance activity-based collaborative learning, allow authentic data collection and modeling tasks, support guided-discovery curricular materials, address learning difficulties identified

through education research, and work in actual classrooms and laboratories with a wide range of students.

Figure 1. MMCE Linear Function. Part I: Coefficient is changed. Pick appropriate graph. Part II: Identify graphs where the chosen coefficient is positive, negative, or zero.

Consider the equation shown below and the graphs displayed just above. Take A to be the graph of the standard conditions. Each question describes a change in one of the coefficients e or f . t represents time. Pick the graph of v above that would best represent the equation after the changes described. You may think of v as representing velocity if you wish. Choose J and explain if no graph is suitable.

$$v = e t + f$$

Compare all changes to the conditions that produced the standard graph A. (not to the previous problem)

____ 13. The magnitude (absolute value) of e is increased (the only change). Which graph would now best represent the equation?

____ 14. The magnitude (absolute value) of f is increased (the only change). Which graph would now best represent the equation?

____ 15. e is set to zero (the only change). Which graph would now best represent the equation?

____ 16. e is set to zero and f becomes a negative number. Which graph would now best represent the equation?

____ 17. Only the sign of e is changed. Which graph would now best represent the equation?

Answer the following questions using graphs A through I. If no graph is correct, choose J.

_____ 18. List any graph(s) where e has a negative value.

_____ 19. List any graph(s) where f has a positive non-zero value.

_____ 20. List any graph(s) where e is zero.

The primary focus of this paper is on necessary pre- or co-requisite student knowledge needed to understand modeling and use it to solve problems. Some of these “necessary” understandings are:

- An understanding of the behavior of the physical world in the topic area (preferably through experimentation)
- Conceptual understanding of the topic area
- Mathematical skills and knowledge
 - Recognize equations as functional relationships
 - Rates of change
- Understanding depictions and representations (overlaps with mathematical)
 - pictorial

- graphical
- vector
- Specific modeling techniques and tool use
- Solving modeling “problems” and general problem solving are necessary for students to understand physics even without a modeling orientation. In some instances it is possible for students to gain the mathematical understanding while doing modeling tasks. An example is given below where students learn to recognize equations as functional relationships while doing analytic modeling in Workshop Physics.

How might we measure student mathematical understandings and improve them?

I am not actually able to tell you precisely how and what students understand. However, I can present a reasonable model that displays some important features about particular topics. In some sense we are modeling student knowledge about modeling. I have picked two examples to discuss in this paper. From the category “Mathematical skills and knowledge” we will look at how well students recognize equations as functional relationships and from the category “Depictions and Representations” we will look at student knowledge of vectors. In both cases we will also examine ways to improve student knowledge.

Equations as functional relationships

The author and Priscilla Laws developed the *Mathematical Modeling Conceptual Evaluation I (MMCE-I)* to measure student understanding of equations as functional relationships. Other parts measure understanding of rates and understanding of vectors which is addressed below.

We will look at two parts of the *MMCE-I* that evaluate functional understanding: the first addresses knowledge of linear equations and the second quadratic functions.

Some questions on the *MMCE* designed to evaluate knowledge of linear relationships are shown in Figures 1 and 2. In Figure 1 when a coefficient of a standard graph is changed in some way, students must pick an appropriate new graph. In a second set of questions, students must identify graphs where the chosen coefficient is positive, negative, or zero. Figure 2 requires students to describe how coefficient(s) is (are) changed to make one graph into another.

Figure 2. MMCE Linear Function. Identify how coefficient(s) is (are) changed to make one graph into another. (There are three more questions of this style.)

For questions 21-25 refer to the graphs and equation $v = e t + f$ (same as those on the previous page) and describe how the coefficients e and f would have to change in order to make one graph into another. Check **one** box for each coefficient that describes how the coefficient changes from one graph to another.

21. How do you change the coefficients to make graph A into E?

(A) *A is standard graph.*

to

(E)

	Becomes or remains zero	Remains the same but is not zero	Increases in magnitude but does not become zero	decreases in magnitude but does not become zero	only changes sign
coefficient e					
coefficient f					

22. How do you change the coefficients to make graph H into G?

(H)

to

(G)

	Becomes or remains zero	Remains the same but is not zero	Increases in magnitude but does not become zero	decreases in magnitude but does not become zero	only changes sign
coefficient e					
coefficient f					

Figure 3. MMCE Quadratic Function. A physical change is made. Identify how the coefficients will change.

$$x = d t^2 + e t + f$$

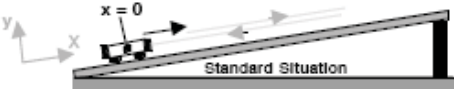
Change(s) made to the standard situation.	I. Which graph describes the new motion?	II. Which coefficient(s) increase in magnitude?	III. Which coefficient(s) decrease in magnitude?	IV. Which coefficient(s) remain the same?	V. Which coefficients become or remain zero?
1. The slope of the ramp is increased so that it is steeper.					
2. The cart is given a harder initial push up the steeper ramp.					
3. The ramp is returned to the original slope and the cart is started further up the ramp at $t=0$.					
4. The clock is started after the cart has already begun moving up the ramp.					
5. The ramp is placed in a horizontal (level) position.					

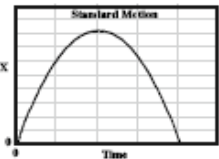
Those designed to evaluate knowledge of quadratic functions are shown in Figures 3-5. Note in Figure 3 that quadratic function is contextualized while the linear function was not contextualized. We originally tried to use a de-contextualized context for quadratic equations but even physics professors had considerable trouble. The contextualized context works well for evaluation of equations as functions. Figure 3 shows questions where physical change is made to the cart and ramp system. Students

must identify how the coefficients will change. Figure 4 shows questions where the coefficients of a standard parabola are changed and students must identify an appropriate graph. Figure 5 shows questions that ask students to identify coefficients related to a particular physical situation. Students must identify an appropriate graph. Figure 5 shows questions that ask students to identify coefficients related to a particular physical situation.

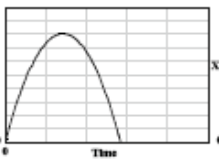
Figure 4. MMCE Quadratic Function. A coefficient is changed. Identify which graph is now correct.

$$x = d t^2 + e t + f$$

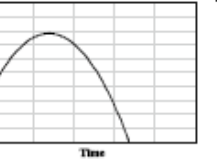





A



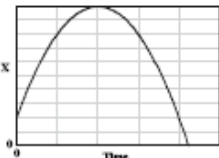
B



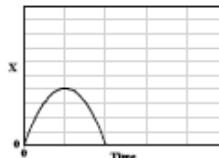
C



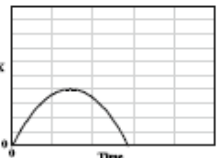
D



E



F



G

J (no graph is correct)

___6. The magnitude (absolute value) of d is increased from that which made graph A (nothing else is changed). Which graph would now best represent the equation?

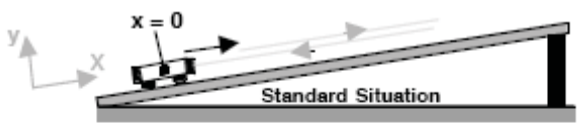
___7. The magnitude (absolute value) of e is increased from that which made graph A (nothing else is changed). Which graph would now best represent the equation?

___8. The magnitude (absolute value) of f is increased from that which made graph A (nothing else is changed). Which graph would now best represent the equation?

___9. The sign of d is changed from that which made graph A (nothing else is changed). Which graph would now best represent the equation?

Figure 5. MMCE Quadratic Function. How are coefficients related to the physical situation?

$$x = d t^2 + e t + f$$



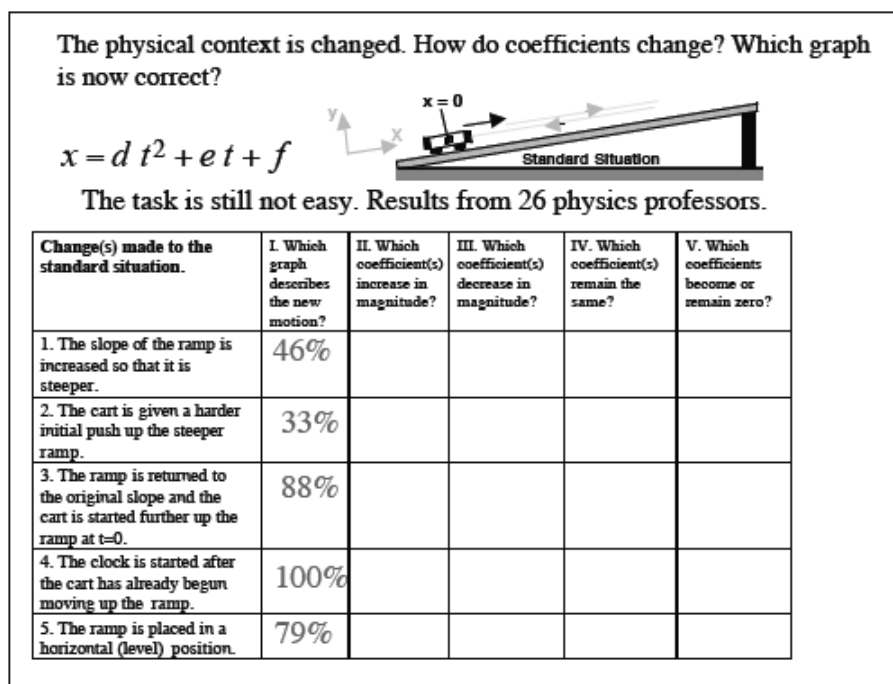
Answer the following about the coefficients d , e , and f .

___10. Which coefficient is most closely related to the slope of the ramp?

___11. Which coefficient is most closely related to the strength of the initial push?

___12. Which coefficient is most closely related to the position when $t=0$?

Figure 6. Result of 26 US physics professors who participated in a modeling workshop answering some of the questions in Figure 4.



How well can professors and students answer these questions?

Figure 6 shows the percentage of 26 physics professors in the US who identified the appropriate graph. (Graph choices shown in Figure 4.) The results are surprising and they certainly indicate this is not a trivial task. How do students do? First let us look at results for non-majors after traditional instruction in a university introductory non-calculus course at a selective university. Figure 7 shows that students after traditional instruction can get about 60% of the linear questions right. The quadratic questions are clearly not known well. While the results did not please us, we did

Figure 7. Percentage of correct answers for questions on the MMCE by algebra-based introductory physics students at a selective US college.

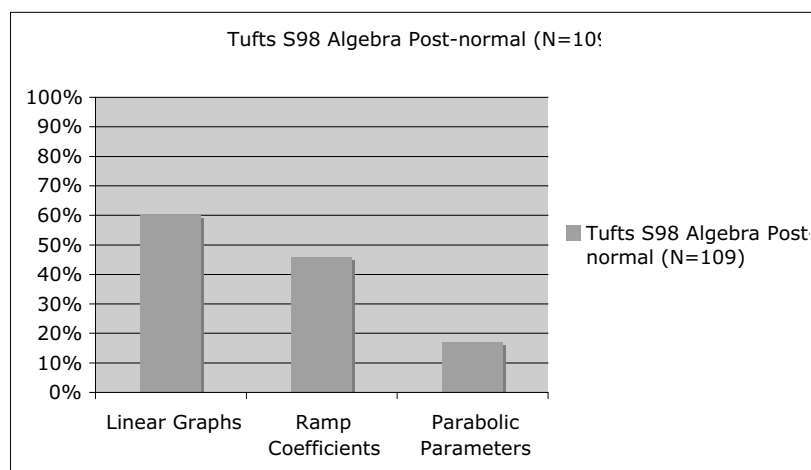
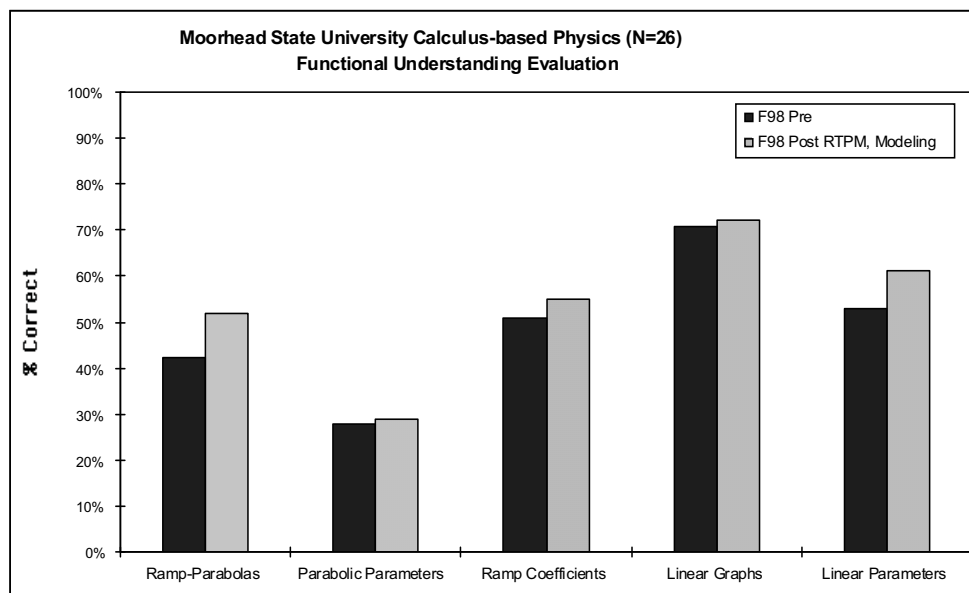


Figure 8. Percentage of correct answers for questions on the MMCE by calculus-based introductory physics students at a US public university. Pre-instruction is compared to results after a curriculum with modeling activities.



have some hope that we could correct the problem with additional instruction. This idea was based on the fact that student scores on conceptual questions in classes such as these after traditional instruction were only in the range of 25% and after introducing *Interactive Lecture Demonstrations* (Sokoloff & Thornton, 2004, Thornton & Sokoloff, 2006) were over 90%. Since more students were able to answer the mathematical questions after traditional instruction we thought it may be easier for the rest to learn it. The problem turned out to be more difficult than we imagined.

Figure 8 shows the percentage of correct answers for questions on the MMCE by calculus-based introductory physics students at a US public university. Pre-instruction is compared to

Figure 9. Percentage of correct answers for linear questions on the MMCE by calculus-based introductory physics students at US institutions.

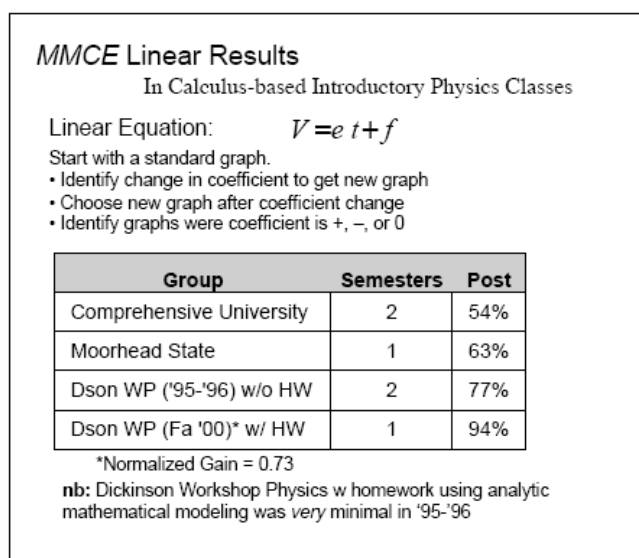
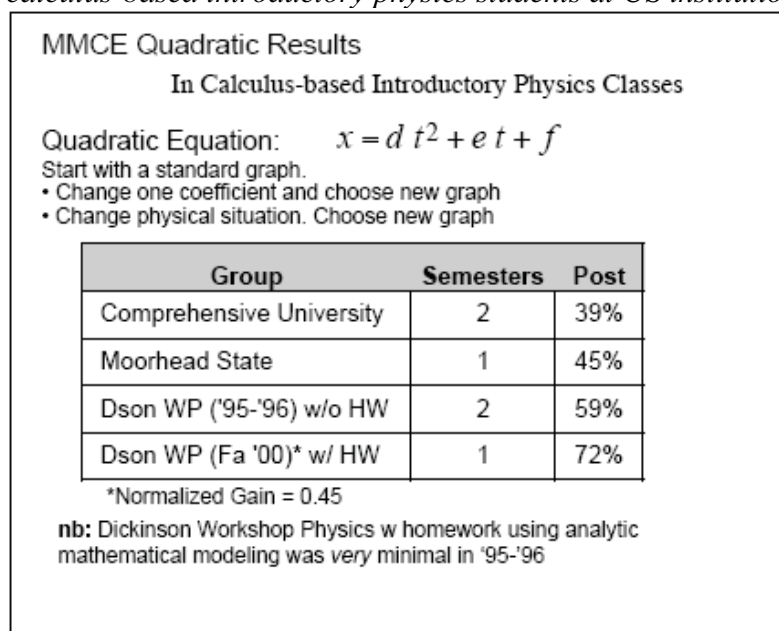


Figure 10. Percentage of correct answers for quadratic questions on the MMCE by calculus-based introductory physics students at US institutions.



results after a curriculum with modeling activities. The results show hardly any improvement in mathematical understanding.

A mild confession and a success story

We have learned to measure reliably student conceptual understandings in both traditional and reform courses. The results of such measures are sometimes not well received by instructors teaching traditional courses.

Figure 11. Modeling use in Workshop Physics.


Units	Topics	In-Class	HW	Functions
1-7	Kinematics & Dynamics	12	8	Linear, Quadratic
8-11	Momentum & Energy	7	7	Linear, Quadratic, Inverse
12-15	SHM, Rotations & Chaos	7	3	Linear, Quadratic, Sinusoidal
16-18	Thermodynamics	7	3	Linear, Inverse, Exponential
19-27	Electricity & Magnetism	8	1	All of Above & Inverse Square
	Total	41	22	

Units	Topics	In-Class	HW	Functions
1-14	Dson Semester 1	22	14	Primarily Linear & Quadratic
15-27	Dson Semester 2	19	8	Also sinusoidal, inverse, Exp

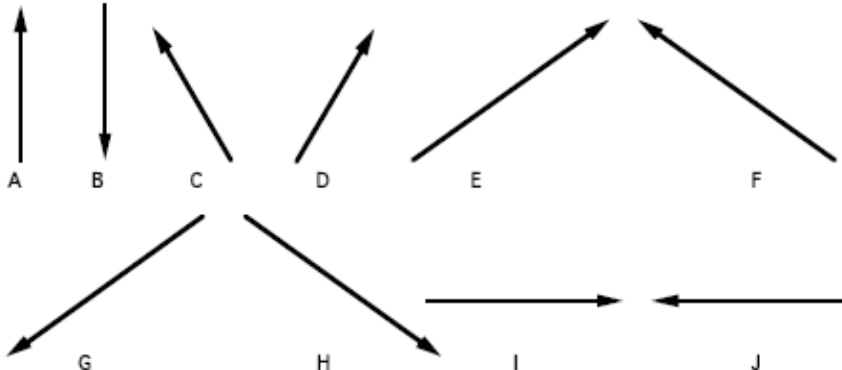
Note: HW use has been approximated for AY 99-00 to present. Modeling Homework was very minimal in 95-96

Figure 12 MMCE II Vector Evaluation-Vector Addition & Subtraction

The following questions are about the two vectors **a** and **b** shown below. (Just as they appear to be the vectors are in the plane (on the surface) of the page.) The vectors **a** and **b** have the same magnitude.



Choose the one answer from the 11 choices A through K below that best answers the question. You may use a choice more than once or not at all but choose only one answer for each blank. If you think that none is correct, answer choice N.



K. The result is 0. N. None of the choices are correct.

- _____ 1. The two vectors **a** and **b** are added using vector addition. Which choice best represents the vector sum $\mathbf{a} + \mathbf{b}$?
- _____ 2. The two vectors **a** and **b** are subtracted using vector subtraction. Which choice best represents the vector difference $\mathbf{a} - \mathbf{b}$?
- _____ 3. The two vectors **b** and **a** are subtracted using vector subtraction. Which choice best represents the vector difference $\mathbf{b} - \mathbf{a}$?
- _____ 4. Which choice best represents the sum of vector **a** with itself $\mathbf{a} + \mathbf{a}$?

After hearing “my students may not know concepts but they understand mathematics” from instructors of traditional physics courses (usually after giving the *FMCE* or *FCI*) we wanted to measure students mathematical knowledge in both traditional and reform instruction and see if students actually understood mathematical concepts. We developed the *MMCE* to serve this purpose. A very successful reform physics program, Workshop Physics (Laws, 2004,1991), places considerable emphasis on conceptual knowledge but also emphasizes mathematical understanding and analytic modeling skills. Figures 10 and 11 compare the results of students in the calculus-based Workshop Physics program with those in students a traditional physics program in a good comprehensive university. The result shows that students in a calculus-based physics course in a comprehensive university only scored 54%(on linear questions) and 39% (on quadratic questions). So it seems that many students in traditional programs not only do not learn concepts well but also do not understand fundamental mathematical concepts. Students in the activity-based Workshop Physics

(who learn concepts and mathematics) scored 77%(on linear questions) and 59%(on quadratic questions) the first time they were given the *MMCE*. After seeing the results, some additional analytic modeling homework was added and the scores have increased to 94%(linear) and 72%(quadratic). These results do not come without effort.

Analytic modeling in Workshop Physics

What is analytic modeling? Students use computer data collection, graphing, and analysis software (e.g. LoggerPro, Coach) to make a visual comparison of data with an analytic function suggested by a mathematical model. They manipulate the parameters manually which enhances their understanding of the meaning of each of the model's parameters and strengthens the students' ability to relate analytic and graphical representations of functional relationships and physical phenomena. Figure 11 shows the number of instances of analytic modeling done in the activity-based classes and those assigned as homework questions. Some particular instances where analytic modeling is used for analysis include ball toss, a mass on a spring, and coulomb repulsion. In all these cases students collect actual data and analyze it. It is necessary for learning for students to do more than one modeling task with each equation.

Student Understanding of Vectors

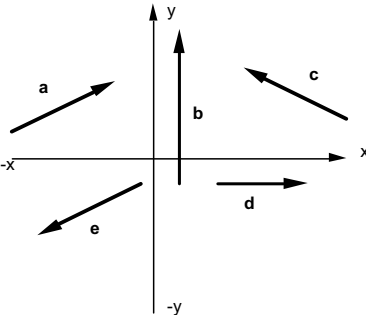
The *Mathematical Modeling Conceptual Evaluation II (MMCE II)* was designed to measure students' conceptual understanding of vectors just as the *MMCE I* measured students' understanding of mathematical functions in the context of modeling. The *MMCE II* measure student knowledge about:

- Vector addition and subtraction
- Components
- Vector Change
- Axis Rotation
- Scalar and vector products

Figures 12 and 13 show the first two categories. All vector questions are context independent. It might not be a surprise that traditional physics courses are largely ineffective in improving student understanding of vectors. Student understanding of vectors as measured using part of the *MMCE-II* showed less than a six percent improvement before and after standard instruction in the Tufts introductory physics class. Less than half of the students understood vector concepts. However, student understanding of vectors before instruction gave us false hope, just as happened with the understanding of functions. In this same class students' conceptual knowledge of Newton's Laws started at 10-15% and traditional instruction had little effect, yet our activity-based curricula resulted in 90% post-test scores (see previous section). Therefore, we

thought starting near 40% for vector knowledge would make further gains easier than our previous conceptual gains. We were wrong again.

Figure 13. MMCE II Vector Evaluation-Vector Components



5. Which vector(s) has(hav e) a non -zero x component ?

6. Which vector(s) has(have) an x component pointing in the negative x-coordinate direction?

7. Which vector(s) has(have) a zero y component?

8. Which vector(s) has(hav e) a non -zero y component?

9. Which vector(s) has(hav e) a y component pointing in positive y-coordinate direction?

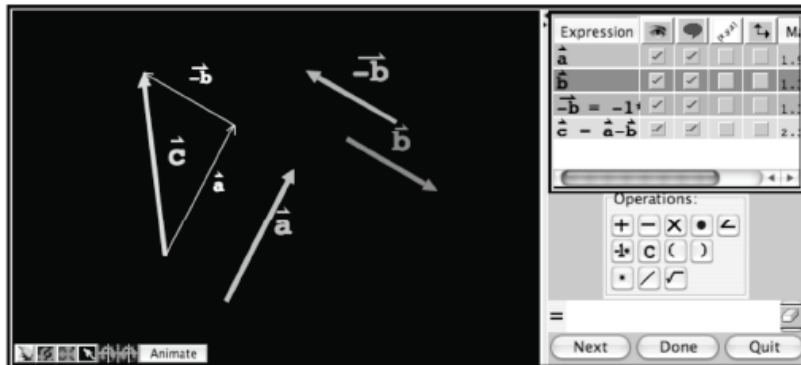
10. Which vector(s) has(have) a zero x component?

11. Which vector has the greatest magnitude ?

We used a vector *Interactive Lecture Demonstration* in class (about 30 minutes) and we assigned a web-delivered vector dynamic tutorial (4) as homework. Both use the *Visualizer*® The *Visualizer*® can display physical data or the output of models in 3-D vector form, including time evolution and trajectories. It understands vector operations and can display 3D vectors graphically and algebraically. Users can change

Figure 14. Web-delivered interactive tutorial with the Visualizer® showing in the window

Vector Subtraction



Vectors can also be subtracted and multiplied. Subtracting \mathbf{b} from \mathbf{a} is the same as adding $-\mathbf{b}$ to \mathbf{a} . ($-\mathbf{b}$ is a vector in the opposite direction from \mathbf{b} but equal in magnitude (length)).

In this example \mathbf{c} is defined as $\mathbf{a} - \mathbf{b}$. Both \mathbf{b} and $-\mathbf{b}$ are shown.

Click the animate button to see the ghost vectors form the sum $\mathbf{a} + (-\mathbf{b})$

Change \mathbf{a} and \mathbf{b} and see what happens to the difference.

The vector $-\mathbf{b}$ has been created by multiplying the vector \mathbf{b} by -1 . This is an example of multiplication by a number to create a new vector.

Show that $\mathbf{a} + (-\mathbf{b})$ is the same as $\mathbf{a} - \mathbf{b}$ by translating the vector $-\mathbf{b}$ and \mathbf{a} to show that their sum is also \mathbf{c} .

How is the vector $\mathbf{b} - \mathbf{a}$ related to the vector $\mathbf{a} - \mathbf{b}$ in magnitude and direction? After predicting, check your answer by subtracting \mathbf{a} from \mathbf{b} in the algebra window and comparing the result with \mathbf{c} (which is $\mathbf{a} - \mathbf{b}$).

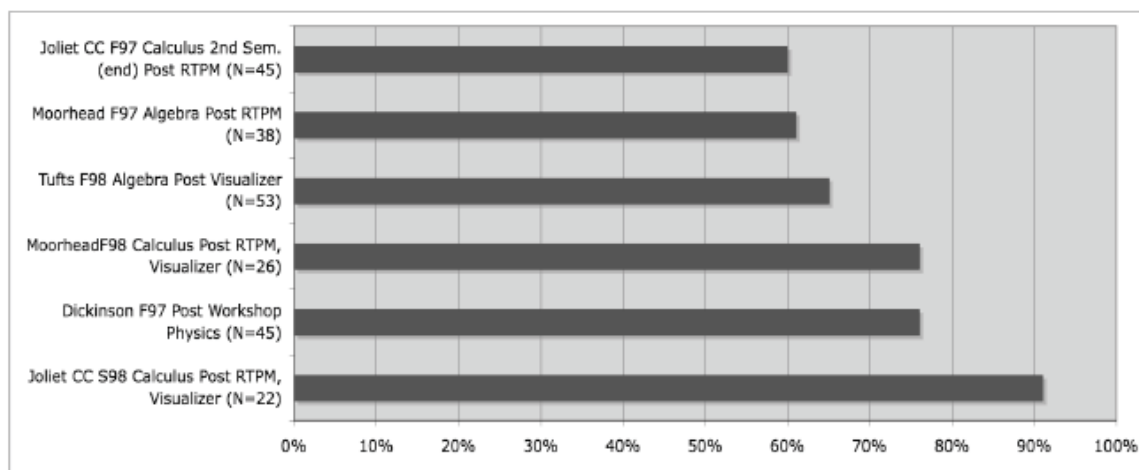
[Next: Vector Changes](#)

[Previous: Vector Sum](#)

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vectors dynamically, observer viewpoint, and coordinate systems. Figure 14 shows and an example of the web-delivered Interactive Vector Tutorial with the *Visualizer*® showing in the window. Students can manipulate the vectors in the window. As a result of using the vector *Interactive Lecture Demo* and the web- vectors in the window. As a result of using the vector *Interactive Lecture Demo* and the web-delivered Interactive Vector Tutorial as homework, we achieved the results shown in Figure15 at Tufts University where 60% of students who did not know the vector concepts learned them. The results are certainly acceptable, but still not what we would wish. Modifications have improved results and results are getting better as shown by the other normalized gains in Figure 15 for very different groups of students. Notice again that reform curricula such as RealTime Physics (Sokoloff, Thornton & Laws, 2004) and Workshop Physics result in more learning of vector concepts than traditional instruction where the improvement is extremely small. The most successful combination for teaching vector understanding was at Joliet Community College where RealTime Physics mechanics and the Interactive Vector Tutorial were used. Students answered over 90% of the questions correctly. It is also clear that computer visualizations can work reasonably well in the curricular contexts we have been discussing.

Figure 15. Normalized gains using part of the MMCE-II to evaluate knowledge about vectors. Students experienced various instruction as noted. Students taking RealTime Physics mechanics and Workshop Physics make substantial gains in vector knowledge unlike standard instruction.



Acknowledgments

I would like to thank Priscilla Laws of Dickinson College for her part in this work.

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