What is the action model? Introducing and modeling principles of least action.

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Abstract
The Action Principle predicts motion using the scalars energy and time, entirely avoiding vectors and differential equations of motion. Action is the tool of choice when we want to specify both initial and final conditions. Maupertuis-Euler action finds the trajectory when initial and final positions are prescribed in advance, but requires that energy be a constant of the motion. Hamilton action finds the worldline when initial and final events are prescribed in advance and easily describes motion when potential energy is a function of time as well as position. A simple toolkit of motion tells us when to use action, when to use Lagrange's equations, and when we must return to the vector methods of Newton. The original Euler method of handling action also provides a basis for computer modeling. Interactive software allows students to employ basic concepts of the principle of least action and increase conceptual understanding.

Least action approach in teaching

The least action principle approach, so important for modern physics, is widely considered to be a difficult topic and is usually only used in advanced mechanics textbooks and courses. Why does it seem a peculiar way to introduce and teach classical mechanics? Why is action as a physical quantity understood as being very abstract and unsuitable for introductory physics, despite the fact that it is a scalar very similar to energy – one of the central concepts of introductory courses?

The reasons are that in the majority of standard advanced texts [like Landau & Lifschitz 1976, Goldstein et al 2002, Marion & Thornton 2003, Hand & Finch 1998], — (1) the mathematics used is *the calculus of variations*, which is not part of the common mathematical toolkit acquired at introductory college level; (2) action is usually introduced extremely briefly and is only used for a quick variational derivation of Lagrange’s equations; (3) action is not immediately illustrated by examples; texts typically include no or a very few examples; (4) there is also no study of the properties of action after introducing it, since they are taught at the end of courses and texts, (5) and finally you find no computer modeling, which means that students cannot obtain direct experience and intuition.

So the important question is how to introduce least action principles? Our experience says that it is possible in the frame of introductory courses provided that we concentrate on the following crucial issues:

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- *Starting with one dimensional cases and using a powerful graphical language*. More general cases only bring in more complicated mathematical expressions, but essentially nothing new in physics ideas
- *Using concrete, but easily generalizable examples.* We have chosen Newton's falling apple, but the arguments will work for any reasonable potential energy function.
- *Building a clear connection to Newton's laws* in terms of comparison.
- *Using* ingeniously *simple original arguments* of the greatest physicists and mathematicians of all times: (1) *Newton's* argument from his celebrated *Mathematical Principles of Natural Philosophy* (1687) and (2) *Euler's* argument from his pioneering work on the variational calculus *The method of finding curved lines enjoying properties of maximum or minimum* (1744). As result we will not need advanced mathematics (all arguments require only high school algebra such as expressions $(a \pm b)^2$ and basic properties of parabola). Moreover we also obtain an excellent foundation for computer modeling, which is important in getting good intuition and experience.

So how does classical mechanics explain the motion of a falling apple? We will show three different approaches – tools for answering this question: *Newton's laws*, *Hamilton's* and *Maupertuis'* principle of least action. As we will see below, together they form a simple toolkit of mechanics in which the question being asked about any system determines directly which tool should be used to predict the motion of that system.

Motion of a falling apple from different points of views

Newton's laws of motion

We start with the well-known Newton's laws of motion, which are already taught at high schools. Firstly Newton says: "Give me the initial state of the apple, which means the initial position and velocity of the apple".

However giving the initial velocity and position means experimentally measuring two nearby positions at very close instants. In this case the initial (and indeed any) velocity is graphically nothing else than the slope of the *position vs. time graph,* or in the language of spacetime physics the slope of the apple's *worldline*.

Then Newton offers us his laws of motion and answers the questions: *What happens next with the apple? That is, what is the position of the apple at the next instant, if there is Earth's gravity or in general some force F (see fig.1)?*

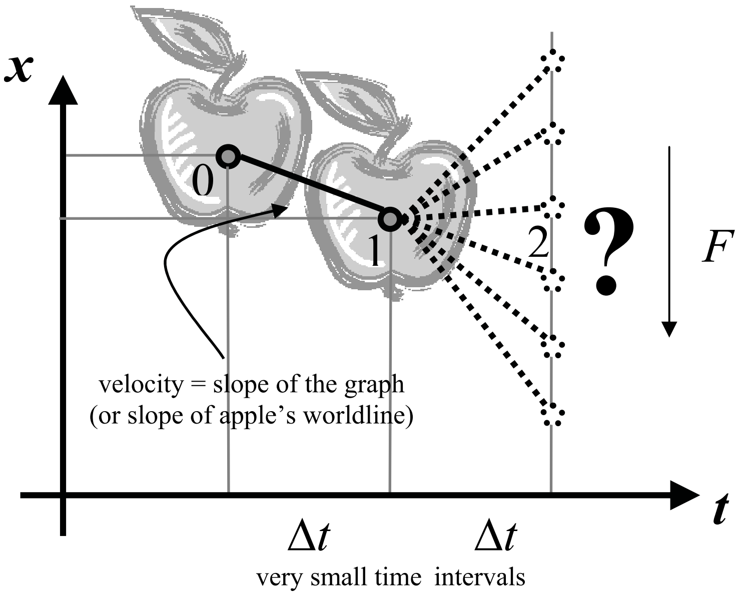

Figure 1. Newton's laws answer the question what is the position of the apple at the next instant, if we know the initial velocity and position or in other words, we know two very nearby positions of the apple.

If there were no acting force, then according to the first law of motion, the principle of inertia, the apple would continue in motion at the same velocity, so graphically it would follow a straight-line worldline (see fig.2). Instead Earth's gravity (or generally some force F) causes a component of motion in the direction of the applied force, as described by the second law of motion, the momentum principle $m\Delta v = F\Delta t$. Competition between these two tendencies results in the parallelogram of which the diagonal represents the worldline of the actual motion (fig.2).

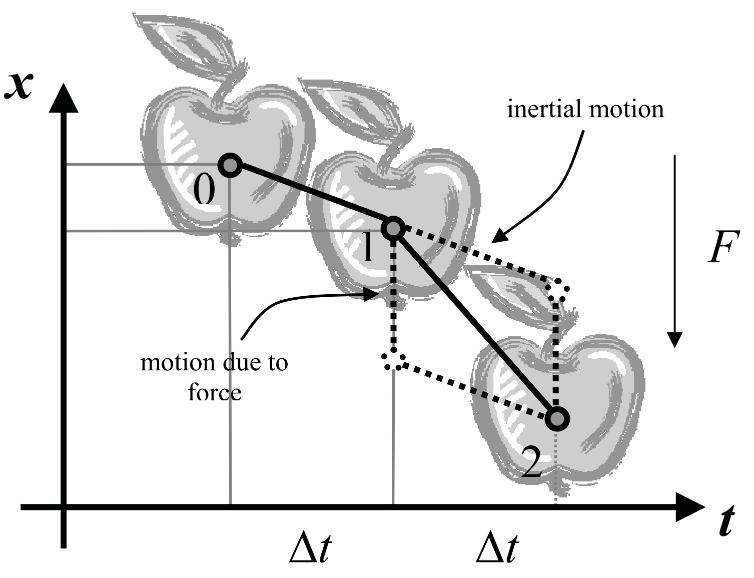

Figure 2. According to Newton's laws the motion of the apple is produced by two "effects": the apple's inertial motion at constant velocity, if no forces act upon it and the motion due to an acting force F. The net motion is then given by the diagonal of the parallelogram of the separate motions that would have occurred.

This process of constructing worldlines (which is conceptually the same for trajectories) is simple, repetitive and universally applicable, so it provides a first-rate foundation for computer modeling. Since today's computers are very fast there is really no need for fancy algorithms in introductory physics teaching. To get a better approximation to the actual motion we simply take smaller time steps.

This numerical method appears in some classics physics texts notably in Chapter 9 in Vol.1 of Feynman's lectures on physics (1964). But it is also very effectively applied in the modern introductory physics curriculum, e.g. in *Modern Mechanics* of Chabay and Sherwood (2002) or in *Unit N* of Moore's Six Ideas That Shaped Physics (2003).

Hamilton's principle of least action

Now we apply a first action model to our falling apple based on energy concepts and Hamilton's least action principle. Hamilton tells us: "Give me both initial and final positions and times of the apple," called in spacetime physics *events*. If we specify the initial and final events in advance, then Hamilton's principle can successfully answer the following question: *What is the middle event for the apple? Or which worldline is followed by the falling apple between the initial and final events, if the apple has potential energy U(x) (see fig.3)?*

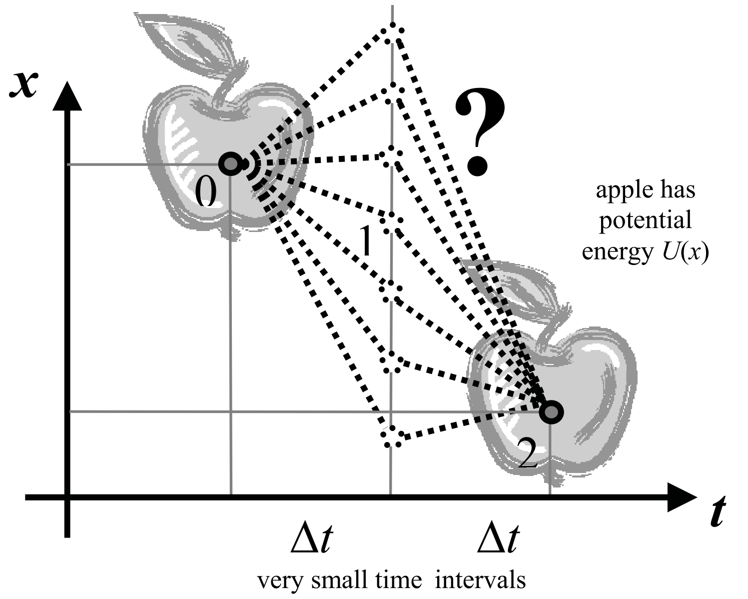

Figure 3. Hamilton's principle answers the question what is the middle position or more generally the middle event for the apple, if we know the apple's initial and final events.

Now what special property does the actual worldline obey? The principle of least action discovered by Hamilton says that the apple follows *the worldline for which the average kinetic energy minus the average potential energy is as little as possible* or put more briefly worldline has *the least action*, because Hamilton's action S is defined as

$$S \equiv \begin{pmatrix} \text{difference between} \\ \text{average kinetic and potential} \\ \text{energy along the worldine} \end{pmatrix} \cdot \begin{pmatrix} \text{time duration} \\ \text{of motion} \end{pmatrix} \quad \text{or}$$

$$S = \left(\langle K \rangle - \langle U \rangle\right) \cdot (t_{final} - t_{initial}) \qquad (1)$$

Using integral calculus the definition (1) has the form

$$S = \int_{t_{initial}}^{t_{final}} (K - U)dt = \int_{t_{initial}}^{t_{final}} Ldt\,, \qquad (2)$$

where the difference $K - U$ is called the *Lagrangian L*, the quantity that appears in Lagrange's equations of motion.

In the case of our falling apple (and also in general case for small Δt) it is easy to calculate all terms in the expression (1) for action S along any worldline 012 (fig. 3). The time duration $t_{final} - t_{initial}$ equals $2\Delta t$. The average kinetic energy $\langle K \rangle$ is given by $(K_A + K_B)/2$, the average of kinetic energies for the first and second segment of the worldline, that is, by $(1/2)(mv_A^2/2 + mv_B^2/2)$, where $v_A = (x_1 - x_0)/\Delta t$ and $v_B = (x_2 - x_1)/\Delta t$.

We now have to pay attention to the potential energy $U(x)$. The shortness of Δt allows us to approximate $U(x)$ by a linear function Cx in the region near point 1. (An additive constant is not important, because it is always zero after an appropriate choice of a reference point.) For the apple constant C is positive and equals mg. Generally we will consider it here as some positive constant. From the viewpoint of the force concept used previously in Newtonian analysis it represents a force $F = -\Delta U/\Delta x = -C\Delta x/\Delta x = -C$, a force in the downward direction. Then $\langle U \rangle$ equals $(U_A + U_B)/2 = (1/2)[C(x_0 + x_1)/2 + C(x_1 + x_2)/2]$. Since the events 0 and 2 are fixed and only position of the middle event 1 is variable, the apple's action S must be only a function of x_1, in which case it is a quadratic function.

To find a worldline with the least action therefore means that we must vary and find a position x_1 which makes the action a minimum. There are two natural ways to do this. One is the trial-and-error method, perfectly suited for a computer which can quickly calculate and compare the action (1) for millions of worldlines. The detailed description of computer modeling based on the so-called Euler variational method is described in our symposium contributions *Action on Stage* (see fig. 1, 2) and *Use, Abuse, and Unjustified Neglect of the Action principle* (see fig. 1).

The second way to find the least action worldline is the use of mathematical methods. According to Hamilton's principle the action has to become larger, if we change the position x_1 of the middle event 1 of the actual worldline by any small displacement δ*x*. Using only high school algebra one can obtain the following expression for the corresponding change in action:

$$\delta S = S(x_1 + \delta x) - S(x_1) = (m\Delta v - F\Delta t)\delta x + 2\frac{m}{\Delta t}(\delta x)^2 \qquad (3)$$

Mathematically equation (3) represents a simple quadratic function with respect to δ*x* whose graph is a parabola. The graphical method proves that the least-action worldline is identical with the worldline predicted by Newton's laws (fig. 4). The method gives students an intuitive and visual understanding of the meaning of the least action principle, as does the computer modeling described earlier.

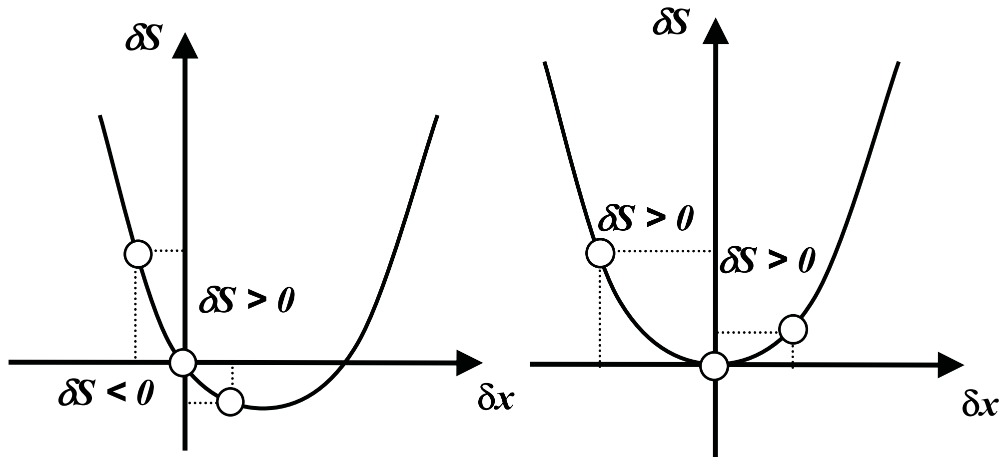

Figure 4. Both parts of the figure display changes in action with respect to displacement δx. In the left part the linear term in eq. (3) is not zero, i.e. mΔv–FΔt ≠ 0. The action demonstrates both negative and positive changes, so the chosen worldline does not yield a minimal action. In the right part the condition that the linear term be zero, mΔv–FΔt = 0, gives a required minimum.

Maupertuis' principle of least action

Finally we will analyze the apple's motion from the viewpoint of the second least action principle called Maupertuis' principle of least action. Maupertuis requires: "Consider a conservative system. Give initial and final position and total energy of the system."

The total energy and its conservation (we again assume knowledge of the potential energy), lead to knowing the apple's initial speed, which implies two possibilities of motion – with the upward or downward direction. But in the case of the falling apple we are interested only in the downward motion. Then according to Maupertuis we are able to answer the question: "*What is the apple's final event, if we know its initial and final positions (see fig. 5)?"*

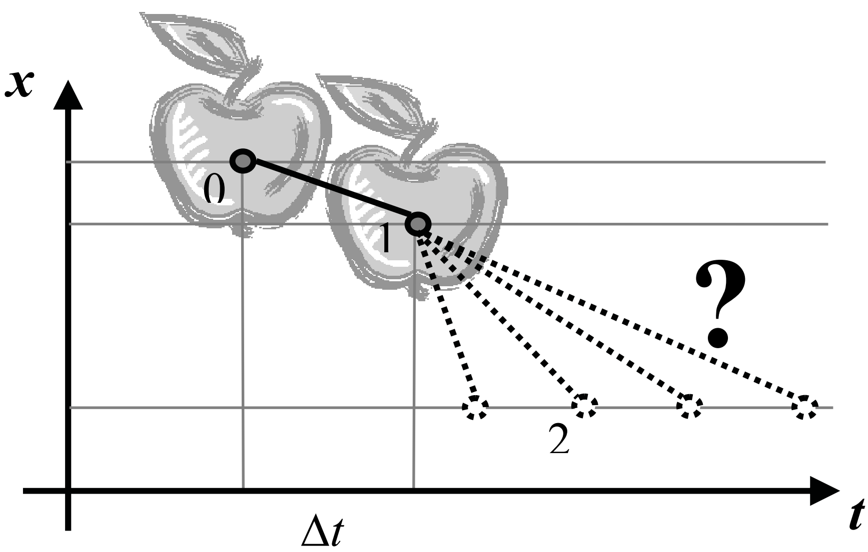

Figure 5. The Maupertuis action principle can answer the question "What is the apple's final event, if we know its initial and final positions?"

Since we consider a conservative system the actual motion of the apple must satisfy energy conservation. From Newtonian mechanics we know that it is the same motion as predicted by Newton's laws. Everything seems to be good. So the natural question arises: where is the action principle? But we now see that we did not realize that energy conservation alone actually allows other worldlines, strange and unrealistic with respect to Newton's laws. One example is shown in fig. 6. How to recognize a motion as actual or unrealistic?

The criterion is just the Maupertuis action. It can be shown in a very similar way as in Hanc et al. 2005 that a useful graphical tool in the case is the velocity vs. position diagram called in mechanics *the phase diagram*, which says that for unrealistic worldlines the area under *the phase curve* is always bigger than for the actual one. This geometric idea provides a foundation for the definition of the second version of action, Maupertuis action W:

$$W \equiv \begin{pmatrix} \text{area under} \\ \text{phase curve} \end{pmatrix} \cdot \begin{pmatrix} \text{object's} \\ \text{mass} \end{pmatrix} \quad \text{or} \quad W = \int_{\substack{\text{initial} \\ \text{position}}}^{\substack{\text{final} \\ \text{position}}} mv\, ds \qquad (4)$$

Summarizing we can say that Maupertuis' principle of least action tells the falling apple to move so that the product of mass and area under the phase curve has the smallest possible value (subject to energy conservation). So far as computer modeling is concerned, it is the problem as before.

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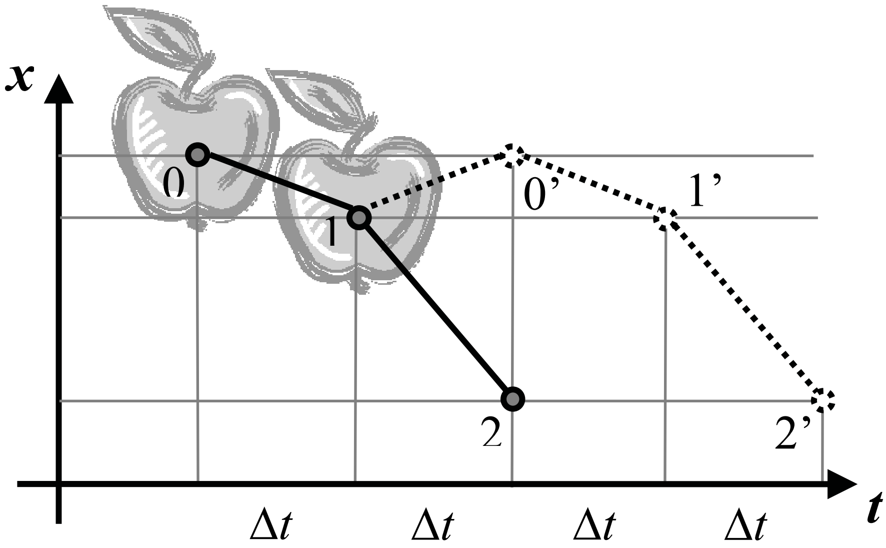

Figure 6. Both depicted worldlines 010 and 010’1’2’ satisfy energy conservation, but only one describes the real motion.

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