The Coefficient Of Restitution Model: How Realistic Is It?

John O'Riordan, Colm O'Sullivan, Patrick Twomey & Michel Vandyck
Department of Physics, National University of Ireland Cork, Cork, Ireland

Abstract

The concept of a 'coefficient of restitution' is often introduced in the context of a model describing inelastic collisions. In this paper we attempt a critical assessment of the model. Some computer-based experiments will be described which help explain the concepts underlying the model.

Introduction

In another paper at this conference (O'Sullivan 2006) one of us has argued that physics teaching could benefit if teachers were to articulate more explicitly the nature of the particular model underlying each topic being taught. In particular, it is contended that clear distinctions should be made between models involving fundamental laws of nature (e.g. Newton's laws), those entailing equations of state (e.g. Ohm's law, Boyle's law) and more primitive models designed to describe, often quite crudely, observed macroscopic phenomena (e.g. 'laws of friction'). The experiment described in this paper was designed and developed to assist students to understand the limits to a commonly used model in the last-mentioned category, namely the use of the concept of a *coefficient of restitution* to describe collisions between two bodies.

The coefficient of restitution is usually defined (Synge 1970) as the ratio of the magnitude of the relative velocity of the bodies after collision to that beforehand, that is

$$e = \frac{|\mathbf{v}_2 - \mathbf{v}_1|}{|\mathbf{u}_2 - \mathbf{u}_1|} \,.$$

It can be shown easily (Mansfield 1998) that, in the c.m. frame, the

$$\frac{\Delta E}{E_0} = 1 - e^2$$

fractional energy lost in the collision is given by

where E_0 is the total kinetic energy of the two bodies before the collision[1]. Thus $e = 1$ represents the case of an elastic collision and $e = 0$ that of a totally inelastic collision.

The issue surrounding the underlying model, in this case, centers on the extent to which the quantity e, and hence $1 - e^2$, is a constant. That is to say, to what extent can *e* be treated as a characteristic of all collisions between the two bodies or is it simply a number that characterizes the energy lost in a particular event? In the latter case there is effectively no model involved whereas, if *e* can be treated as a characteristic of the interaction, the model (a *'law of collisions'*?) asserts that the loss of energy is proportional to the initial energy, that is

$$\Delta E \propto E_0.$$

Since there does not appear to be a fundamental *a priori* reason why such a 'law' should apply in real collisions between macroscopic bodies, it was felt that an experiment to investigate the extent of applicability of such a 'law' and related issues would be instructive.

The experiment

The experiment chosen to study a collision process is shown in figure 1. The use of a compressible helical spring to provide an interaction between colliding bodies, which has a sufficiently long impact time to enable the details of a collision to be investigated, is familiar in many schools and universities. Collisions between carts, or between one cart and a fixed object, moving on a track in a low friction environment are particularly suitable for such studies. Because, in the case of our experiment, it was felt that it would be interesting for students to make measurements before, during and after the collision, standard techniques for measuring positions and speeds, such as ultrasound motion sensors or light gates were not sufficiently accurate. Since the cart used has built-in magnets, it was decided to use an appropriately calibrated magnetic field sensor[2] to measure the position of the cart. The force on the fixed end of the spring was measured by a force sensor[3] rigidly fixed to the track and connected to the spring as in figure 1. Since the target is fixed to the track, all observations are in the c.m. frame.

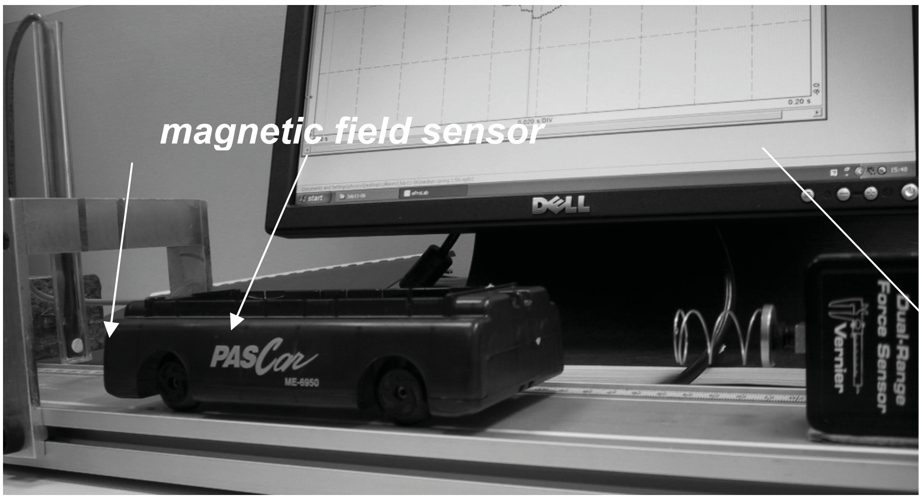


Figure 1: View of the experiment

Both sensors were connected to the e-ProLab data acquisition system developed under the ComLab project4. Any standard data acquisition system may be used for the experiment described but, in contrast to other systems, e-Prolab provides open source information for development of specific applications and can easily accommodate homemade and most commercially available peripherals. The e-ProLab screen for a typical collision event is shown in figure 2. Appropriate data can be cut and pasted into a spreadsheet application and curve-fitting tools used to determine the speeds of the cart before (U) and after (V) the impact with the spring.

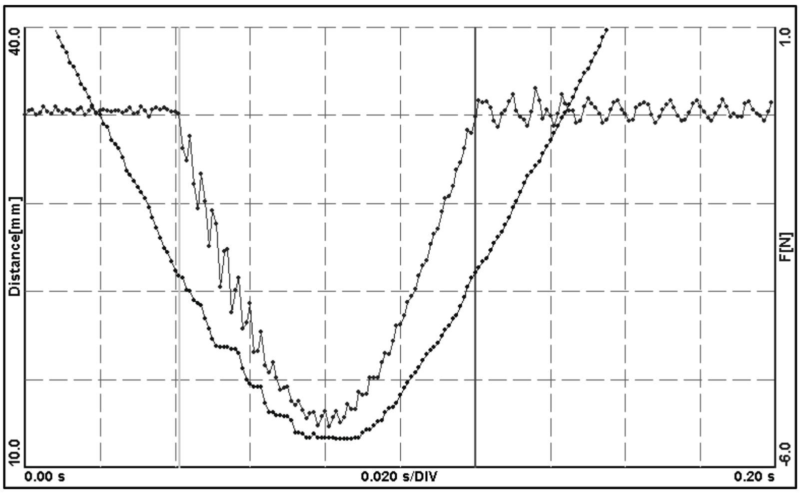


Figure 2: Typical plot of the position of the cart (blue) and reading of the force sensor (red) versus time. The force is negative because a compressional force on the sensor is measured as negative. The impact time of the overall collision is the time between the two vertical markers.

A number of important concepts in analytical mechanics may be investigated by students.

1. To show that the momentum transfer is equal to the integral of force over time

The e-ProLab software allows for the determination of the area under the force–time curve which enables the comparison of the momentum transfer, $M(U + V)$ where M is the mass of the cart, with $\int F dt$. Sample results are shown in table 1.

2. Measurement of the force constant of the spring

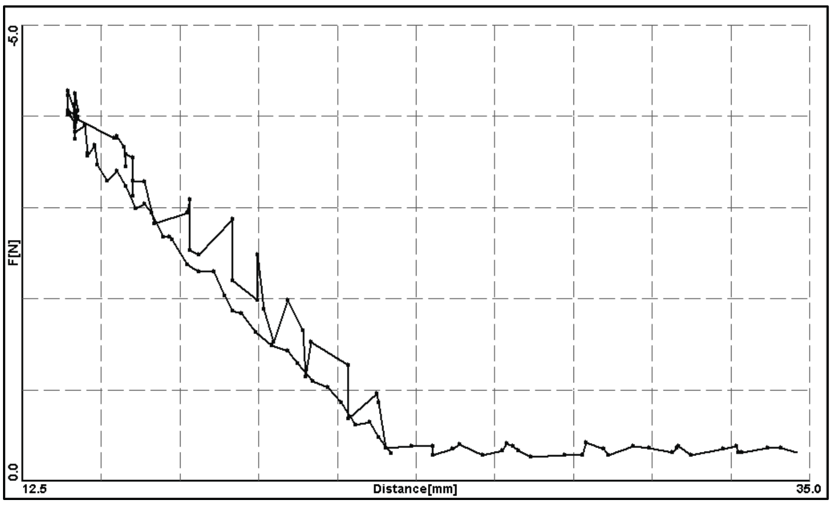
The force constant (k) of the spring may be determined from the data in figure 2 in two ways.

(i) The time of contact, ≈ 80 ms in the case of the data in figure 2, is clearly one half of the period of oscillation of the system comprising the spring with the cart attached. Since this period is equal to $2\pi\sqrt{(M/k)}$, the value of k is easily determined.

run	U (m/s)	V (m/s)	$\int Fdt$ (N s)	$M(U+V)$ (N s)
M 1N	0.085	0.084	0.044	0.042
M 2N	0.167	0.164	0.080	0.082
M 3N	0.268	0.256	0.140	0.130
M 3,5N	0.340	0.327	0.170	0.166
M 4N	0.396	0.330	0.190	0.181
M 5N	0.478	0.465	0.240	0.235
M 6N	0.494	0.494	0.240	0.246
M 7N	0.676	0.560	0.290	0.308
M 8N	0.803	0.783	0.400	0.395
M 10N	0.831	0.798	0.410	0.406

Table 1: Comparison of momentum transfer and $\int Fdt$ *for ten sample collisions*

(ii) The data in figure 2 can be presented in the form of an $F\!-\!x$ plot (figure 3). The slope of the portion of the plot corresponding to the collision, determined for example by exporting the data to a spreadsheet utility, enables the calculation of k.



(iii)

Figure 3: Typical force versus distance plot

Sample results are presented in table 2.

RUN →	W 1,7N	W 1,5N	W 1,3N	W ,90N	W ,60N	W ,50N	W ,35N	W ,17N
***k* from impact time**	58.5	58.5	58.5	58.5	58.5	58.5	61.4	58.5
***k* from *F-x* curve**	57.4	57.1	55.1	60.2	58.0	55.7	54.1	51.0

Table 2: Sample results from the measurement of the force constant of a spring

3. *Energy conversion during impact*

The maximum force (F_{max}) experienced by the spring and the corresponding maximum compression (X_{max}) may be measured from the data in figure 2 and similar plots for other collision events. The corresponding stored potential energies ($\frac{1}{2}kX_{max}^2$ and $F_{max}^2/2k$, respectively) may be compared to the initial kinetic energy of the cart (table 3 below).

RUN →	W 1,9N	W 1,7N	W 1,5N	W 1,3N	W ,90N	W ,60N	W ,50N	W ,35N
kinetic energy before impact = $\frac{1}{2}MU^2$ (mJ)	14.16	14.03	11.49	8.48	6.22	2.98	2.54	1.34
energy of compressed spring = $\frac{1}{2}kX_{max}^2$ (mJ)	13.09	12.19	10.04	6.73	6.28	2.88	2.32	1.16
energy of compressed spring = $F_{max}^2/2k$ (mJ)	14.18	15.48	11.60	8.35	5.92	2.76	2.23	1.10

Table 3: Sample comparison of kinetic and potential energies in eight collisions

4. *Energy lost in a collision*

In all situations involving collisions between a cart and a compressible helical spring the energy loss is small, that is *e* is close to unity. It can be seen, however, that some energy is always lost; energy is always observed in the form of post-collision oscillations as seen clearly in figure 2. This observation provides students with a useful insight into the nature of non-elastic collisions in general. Some amount of energy is also lost as a result of friction between the cart and the track.

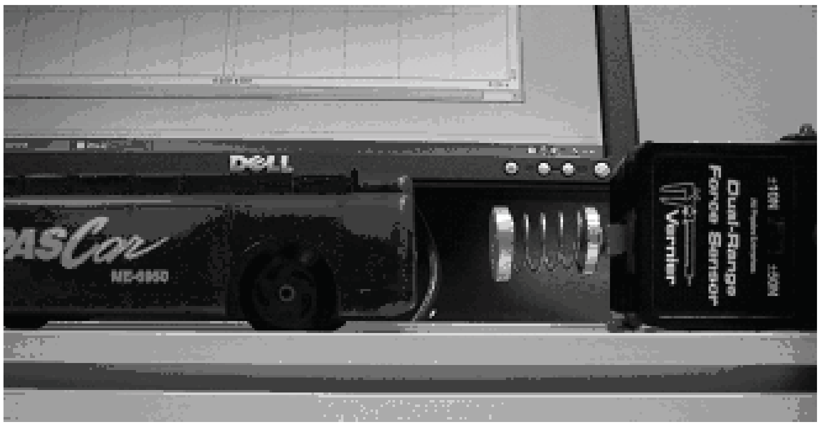


Figure 4: Experimental arrangement showing a mass attached to the spring

It is often more interesting for students if the study of energy loss in collisions is carried out when a mass (~ 10 g) is attached to the end of the spring (figure 4).

In this case there are three principal sources of energy loss, namely

a) Energy lost during the initial impact of the cart with the attached mass. This collision may be assumed to be totally inelastic; that is, $\Delta E = E_0/(1+M/m)$. The issues here are interesting but complex; it is hoped to discuss this question elsewhere.

b) Energy transferred to spring oscillations ($F_m^2/2k$).

c) Energy lost by 'friction' ($\approx$ friction force x $2X_0 = 2(M + m)X_0a_f$, where a_f is the mean acceleration due to friction between cart and track) while the cart is in contact with the spring. A value of the acceleration may be estimated by fitting a quadratic function to the distance–time data recorded.

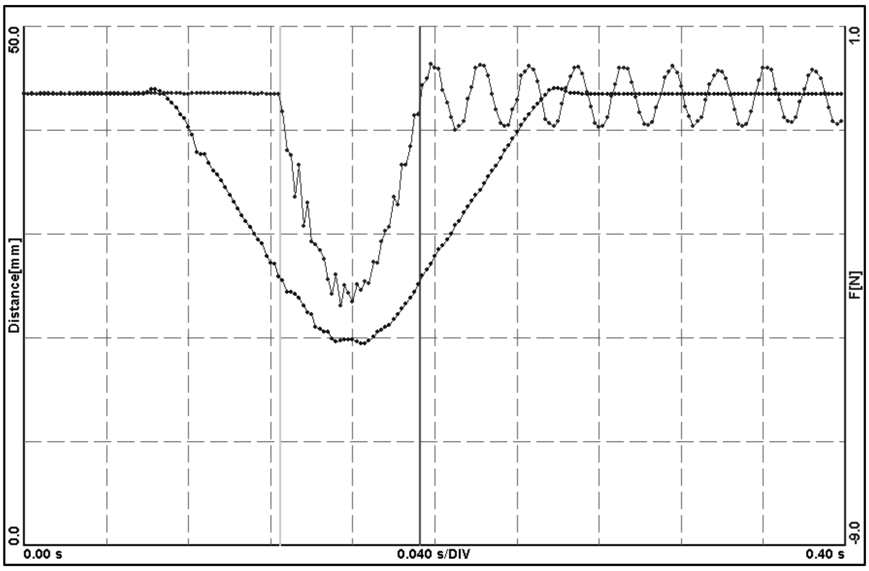


Figure 5:Typical plot of the position of the cart and the reading on the force sensor versus time when a mass is attached to the spring

In this case the energy transferred to oscillations of the mass-spring system ($F_m^2/2k$, where F_m is the amplitude of the force oscillations observed) is much clearer (figure 5) and can be measured more easily. Table 4 shows some sample results from this analysis.

		run 4.1	run 4.2	run 4.3	run 4.4	run 4.5	run 4.6	run 4.7	run 4.8
Incident speed (U)	(m/s)	0.10	0.18	0.25	0.31	0.40	0.48	0.57	0.61
Collision energy ($E_0 = \frac{1}{2}MU^2$)	(mJ)	1.26	3.95	8.13	12.42	19.96	28.91	41.08	46.42
Energy lost in inelastic collision	(mJ)	0.09	0.29	0.60	0.91	1.47	2.13	3.02	3.41
Energy in oscillations	(mJ)	0.02	0.08	0.16	0.30	0.56	0.76	1.06	1.26
Energy lost to friction	(mJ)	0.01	0.02	0.03	0.03	0.05	0.05	0.07	0.06
Total energy loss (ΔE)	(mJ)	0.12	0.39	0.79	1.25	2.11	3.03	4.15	4.73
$\sqrt{(E_0 - \Delta E)/E_0}$		0.952	0.949	0.950	0.948	0.947	0.948	0.948	0.948
$e = V/U$		0.900	0.944	0.961	0.952	0.958	0.955	0.960	0.949

Table 4: Example of audit of energy loss in collision (mass attached to the end of the spring)

5. *Study of the coefficient of restitution*

Values of the coefficient of restitution, measured as the ratio *V/U*, are included in table 4 above. The corresponding energy loss ΔE calculated in the table can be used to determine $\sqrt{\{(E_0 - \Delta E)/E_0\}}$. The results obtained are compared in figure 5; the error bars indicate approximately one standard deviation.

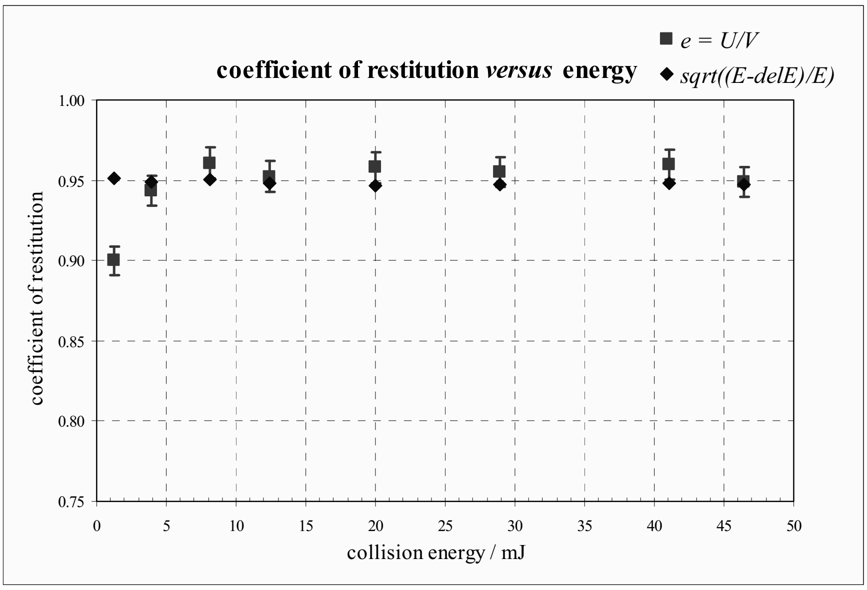


Figure 5: Typical plot of the dependence of the coefficient of restitution on energy

Having discussed these and similar results using springs of different strengths, students may conclude that, within the accuracy of this experiment, the underlying model is applicable to most of the collision events studied but that there may be some systematic variation from the model at lower energies. They may conclude that the applicability of the model requires that the principle sources of energy loss must be proportional to $E_{0,}$ within the accuracy of the experiment. Sources a) and b), but not c), in section 4 above satisfy this requirement. Further considerations may lead to speculation on situations in which the model is unlikely to be applicable and to proposals for further experiments.

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Mansfield, Michael, O'Sullivan, Colm, Understanding Physics, John Wiley & Sons 1998, page 138.

[1] In the laboratory frame $\frac{\Delta E}{E_0} = \frac{m_2}{m_1 + m_2}\left(1 - e^2\right)$ where m_1 and m_2 are the masses of the moving and target particles, respectively. op. cit. page 138.
[2] http://www.vernier.com/probes/mg-bta.html
[3] http://www.vernier.com/probes/dfs-bta.html or http://store.pasco.com/pascostore/showdetl.cfm?&DID=9&Product_ID=1468&Detail=1
[4] For details of the ComLab project see http://e-prolab.com/comlab/