Promoting the competence of mathematical modeling in physics lessons

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Abstract

Mathematics plays a twofold role for physics: first it allows for predicting results and outcomes of experiments; second, it provides a structure for the description of consistent theories.
The predictive power of mathematics for natural processes is in itself fascinating. But most pupils do not like mathematics in physics. However, in everyday life they encounter physical quantities where it may be important to know the numbers as well as the relations between different quantities. Therefore ways are sought to increase the eagerness of pupils to apply mathematics to physics phenomena. Making curious about numbers or showing relations between different processes might be a promising way. For this goal not only exact computations but also qualitative reasoning with aid of some numbers is crucial. Some examples are given.
Ways are discussed how to enhance the abilities of pupils in developing and evaluating mathematical models to physics problems. The use of graphical representations for a connection between experiment, measurement values, graphical representation and mathematical description is considered.

Introduction

The common overall goal of science education is that students learn to appreciate science and know the scientific method as part of the basis for life long learning. The students should gain insight into the scientific building and its internal structure. A quite important part of this building is the use of modeling and mathematics in science, especially in physics.
Most curricula expect at least that the students at the end of their school carreer are able to model physics phenomena. But modeling is by no means restricted to physics. Nowadays the whole of science and even beyond, e.g. to economical questions, cannot be thought of without mathematical modeling: it delivers predictions or provides theoretical models for explaining. The basic aspects of mathematical modeling in complex phenomena gain importance, especially as the computer power increases and more and more problems can be treated numerically in ever increasing precision. Since in physics the process of modeling is most simple students should gain insight into the way mathematics and modeling is applied in physics as an important part of the scientific method. Successful modeling not only requires that students know the properties of a model but also have the ability to make predictions about definite values. This step needs mathematics and especially the ability of translating between the physics phenomena and the mathematical formulation. It seems to be the crucial difficulty and hence needs special attention in design of physics courses.

The role of mathematics in physics

There is a vast amount of literature on this topic. I only give one citation:

Since the times of Newton and Galilei the importance of mathematics for physical problems is undoubted. The application of mathematics tools to physics allowed for the step from philosophy of nature to modern science. The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. (Wigner)

The relation between mathematics and physics is not unambiguous: First of all mathematics is viewed as a tool in physics. It serves as a toolbox in providing functions, equations and rules for manipulations.
The main goal of physics is to make predictions, to determine values of physical quantities and eventually to verify whether the assumptions of an explanation might be correct. The efficient interplay of experiment and theory needs the mathematical description.
One could ask whether it could be possible at school to set aside the mathematics and rely only on a qualitative description. I would say in this connection that if we want to convey physics as a science and not as kind of knowledge about nature we need some mathematization.

Example: It can be observed that an accelerating body which for instance is falling, gets faster and faster. But this description does not give the exact dependence: some values of the fall have to be taken, compared and analysed, as e.g. demonstrated by Wagenschein, who uses strictly mathematical reasoning.

But the role of mathematics goes beyond this seemingly minor part as a tool. The example already hints to the next point:

Mathematics can provide the structures that help in analysing physical phenomena, seeing analogies and promoting physics research. As seen for instance with the Noether Theorems mathematical results can be of immense relevance for the fundamentals of physics. They reveal the inner structure and give hints for possible appearances.

It may be that the development of physics requires new mathematical tools and – on the other hand - that mathematics allows for predictions that have to be verified in the experiment.

Mathematics in some sense is the skeleton of physics: Sometimes qualitative explanations are only easily done, if the mathematical background is known. This structuring role is essential for theories and the derived models.

It may hence not be forgotten that the mathematical tools need interpretation in term of physics: The same mathematical structures may apply to several distinct physical phenomena: e.g. the equation for the electric potential, dependent on the charge distribution, the equation of heat transport and the stokes equation.

Mathematical concepts turn up in entirely unexpected connections. Moreover, they often permit an unexpectedly close and accurate description of the phenomena in these connections. Secondly, just because of this circumstance, and because we do not understand the reasons of their usefulness, we cannot know whether a theory formulated in terms of mathematical concepts is uniquely appropriate.(Wigner)

But also on a quite elementary level interpretation is necessary and consequences from the statements of mathematics have to be considered.

So the development of the relation between mathematics and physics may comprise the following steps:

- Physical phenomena are observed (surely on the basis of hypotheses or with a theory in mind) and they are analysed in a precise experiment or observations.
- They are described: What happens? What could be the relations: the more .... the less or vice versa ?
- An explanation is thought: Why does this happen?
- The influences of different parameters have to be separated and analysed, either experimental or theoretical. Idealisation and then mathematical modeling take place and give numerical predictions.
- These predictions can be tested and may lead to further research.
- The model has to be analysed and interpreted in all its derivations: Are all derivations consistent with the experiments or observations? Are further experiments necessary?

If looked at more sharply it becomes clear that most of these activities require reasoning and talking and communicating.[4] The mathematics part is only at the end of a whole process and as stressed before needs again interpretation: What is the meaning of the structure? Has the equation implications not seen before (e.g. prediction of positron)? Is the model right in the limits (e.g. very high or very low temperature)?

What should pupils know about the role of mathematics in physics?

Mathematics is a fundamental part of the physics building. Hence the introduction to the techniques of mathematization is one important goal especially of physics education at school as the students have to learn to know the scientific method and different ways of gaining knowledge. There are several points to be conveyed:

- mathematics shows analogies between different processes: The formula for the Bremsweg of a braking car $s=\frac{v^2}{2a}$ as well as the formula for the maximal height of throwing $h=\frac{v^2}{2g}$ is treated in school. Students have to learn that the underlying physics is the same which in this case furthers the understanding. Several analogies occur in school physics e.g. for different forms of energy which

4It is told, that Feynman - to the astonishment of some guests - first discussed a matter before he went on to use equations or formulae or any other kind of mathematics are defined in a similar manner: kinetic energy, magnetic energy of a soloid or the energy of a spring. Similar argumantes are used to derive the formulae. Therefore it becomes clear that mathematics only provides the structure. It has *per se* no meaning as mathematical constructs are at first abstract terms. They only get a meaning by interpretation.

- mathematics helps to recognize a structure or understand relations more precise.
- mathematics is more precise than qualitative descriptions. It states not only: the more – the more, but specifies the dependence as linear, quadratic, fourth power, exponential, ...
- mathematics allows for predictions or estimates: the acceleration of a gymnast doing bar exercises, e.g. a big circle, or in a looping, the energy needed by a sportsman,

The aspect of doing estimates is in my view a very important one: The own questions of the students could be treated. They then have to do the modeling process on their own and translate it in known mathematical structures. I will come back to this point later.

What do students think about the role of mathematics in physics?

We did a small pilot study with a questionnaire in order to get some idea before broadening the sample (s. Fig.1). The results of this preliminary questionnaire give some first hints of the students' views (grade 11, 15 students).

They are mostly convinced that mathematics is an important tool of physics. But they seemingly don' think that with mathematics processes could be described. This answer would have to be analysed further with aid of interviews.[5]

The small experience of students with physics laws and a description by formula could also lead to the opinion that formula (or mathematics) could not show similarities. Nevertheless most pupils think they need mathematics to understand physics, but they do not agree that mathematics furthers understandig remarkably. However, it surely contributes to understanding. Here interviews would be needed to explore the meaning of these statements better, especially with respect to the meaning of "understanding physics"[6].

This first glimpse on students opinions seems to show that some important aspects of the role of mathematics in physics are not acknowledged by the students. This may be caused by the big role formulas and rote learning play at school. It has to be taken into account, that most students only know very few formulae by heart beause they mainly use a formulary. Since the students' view is mostly imfluenced by the use of formula, we take a closer look in the students' view about the role formulae in physics (lessons).

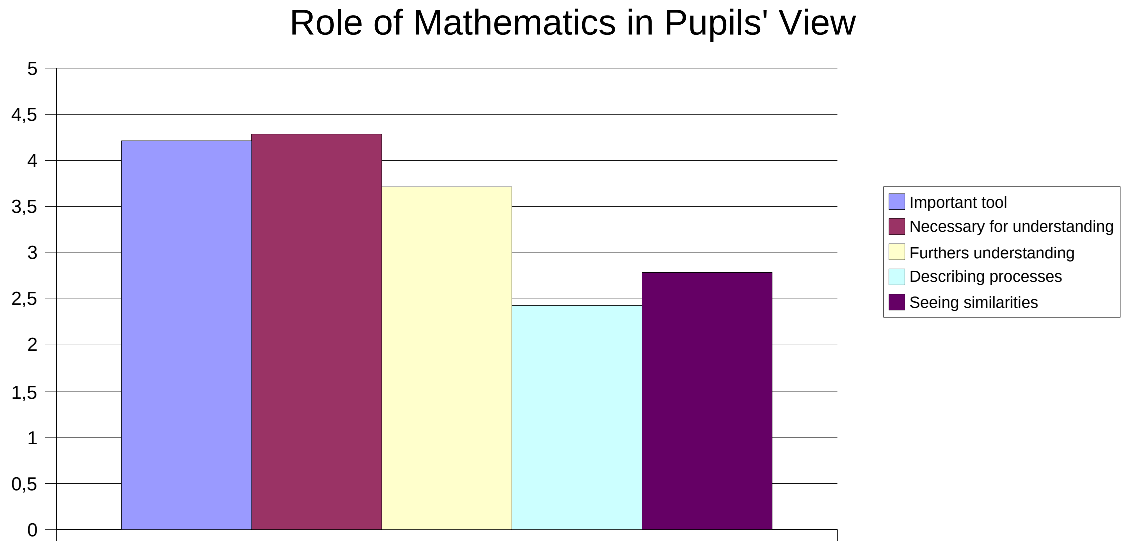


Fig. 1: Mean value of agreement in a class (grade 11) of 15 students. (5: I agree totally, 1: I disagree)

5Furthermore it would be important to learn to know the teachers conceptions and goals about mathematics in physics.

6It seems that some students learn formula by rote and rely on them in exams because they have the feeling not to understand the physics behind.

The role of formula

The students do not mean that they understand a subject better by learning a formula. But it seems, that formulae might give the pupils the certainty and a fixed basis on which to argue.

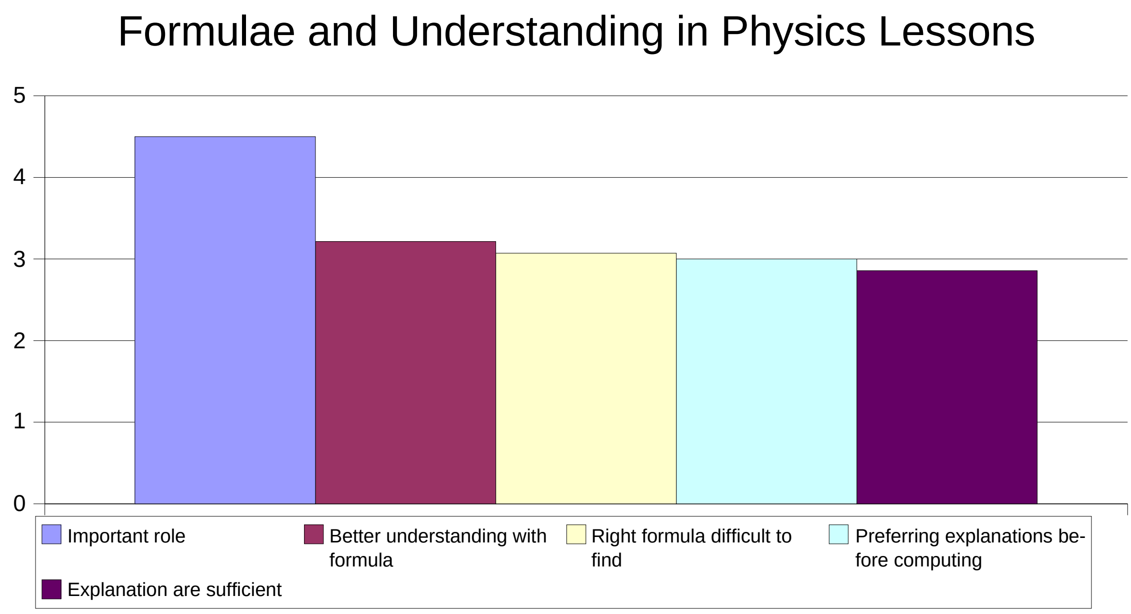


Fig. 2: Mean value of agreement to these statemants concerning the role of formulae in physics lessons in aclass (grade 11) of 15 students. (5: I agree totally, 1: I disagree)

This is supported by the statement: “With formula I understand physics better” where students seem to agree slightly. However, they think explanations won't be sufficient to physics understanding. But throughout the students agree that formulae are an important part of physics.

If they are asked, whether ist is difficult to find the right formula in a given problem, they answer undecided. Mostly the exams or tests are constructed in such a way, that only a few formula are needed and can be guessed from the general subject and the lessons. Also the tabulary helps a lot. Nearly all students use it regularly and very often.

There are some deficiencies in the view of students on the role of mathematics which seem to mirror the emphasis laid in school on calculation. If however the goal is insight of students in the scientific method (s.e.g. Hestenes 1992) and the ability of applying the physics content in everyday life then the emphasis in school lessons has to be shifted to modeling which also broadens the accessible subjects in school.

Which proposals could be made to enhance the ability of modeling?

Proposals for promoting the modeling competence

The ability of modeling requires in the first step that the students can recognize the connections between different physical quantities and concepts and their structure and that they have understood their dependencies. It may not be forgotten that for the application of mathematics in physics the semantics is most important. Therefore the road from observing physics phenomena to mathematical modeling has to be planned carefully and has to be gone slowly and purposefully.

1. Clarifying the way of gaining insight

Many problems students have mainly consist in translating between the physical objects with their relations and the mathematical formalism. To achieve this ability a careful path from the everyday experience to the formula has to be observed. Five steps show important for bridging the gap between the “world of objects” and the “world of formulas”:

1. Activation of own experiences or experiments
2. Qualitative formulation of relations
3. semi-quantitative formulation, together with a graphical representation
4. Introduction of formula
5. Re-interpretation of formula and hints to analogies

Steps 2 and 3 are most important for insight into the process of modeling. They form the ladder from daily life to the abstraction in mathematical formulation. The main goal is the understanding of the process leading to a formula. For this, especially step 5 is essential for anchoring the meaning of a formula, but too often it is neglected. Students tend to simple learn the characters of a formula, hardly remember their meaning and – not interpreting it in their own words – they may just interchange the different chracters: the formula is useless and meaningless to them. But the presupposition for recognising analogies or similarities in physics consists in understanding the physics background, seeing the dynamics and then making purposefully the transition to the mathematical representation.

2. Making curious about mathematics in physics

In order to overcome the common disliking of computational tasks in physics I suggest to bring the students to ask about definite values of physical quantities. Suitable methods – also for furthering physics understanding - would be:

- problems that separate between the task of analysing the physics contents and the computation of numerical values
- differing explanations or hypotheses of the same phenomenon that only can be decided by some computation
- amazing questions that stimulate the students for wanting to know numerical results

It is a difficult task of physics education research to develop kinds of problems that further insight into the structure of physics as well as the ability to translate in simple cases physics problems into a mathematical representation by the process of modeling.
An important aspect herewith is idealisation and simplification. So in modeling real processes students have to recognize which parameters are important and which may safely be neglected (Wells, Hestenes, 1995). But sometimes students do not know when they have to look carefully and when they have to neglect a parameter regarded as important. The reduction to a level appropriate for school is quite important and one of the difficulties in interesting problems..
In simple cases students can do the idealization part of the task with surprising confidence: They are aware (e.g. in mechanics) that often friction is neglected, that a body is regarded as a mass point and so on. Once the students have grasped the significance and characteristics of models (s.e.g. Mikelskis-Seifert, 2005), this insight should be used to open up new subjects for students in physics lessons with a twofold goal: making physics interesting and relevant for daily life and furthering their abilities and selfconfidence. The possibility of treating interesting problems not only in mechanics but also in other subjects as thermodynamics might enhance the motivation or interest also with girls. The training of modeling should be used to treat problems not only with qualitative reasoning - which is extremely important - but in some cases to finish the reasoning by concrete calculations which give the students the certainty to be right, especially with estimates for everyday phenomena.

Examples for simple tasks of mathematical modeling

I will explain two examples in order to make clear that the mathematical formulation is only the end of the modeling process.
Example 1 (Grade 11): Which acceleration an athlete doing a grand circle at the horizontal bar has to endure in his wrists?
Unfolding: How can the man rotate? He uses the vibrations of the reck, and moves his body (stretching and bowing) in order to sustain the movement.
Reasoning and reduction: The vibrations of the bar are too difficult, the bodie's inner movement also, the mass is thought to be accumulated in the center of body, the distance to the bar ist estimated by approximation. Hence the following assumptions have to be made:

- The bar is assumed rigid.
- The friction is neglected.
- The mass is concentrated in the middle of the body.

These three simplifications are sufficient to do the calculation and the result gives $6g$, where g is the earth acceleration.
As simple as this example is, the students could go mainly by themselves the way from everyday life to the physics and the subsequent computation. A training of the students in this spirit is important (Hestenes, 1992), It should be used not only in mechanics but also e.g. in thermodynamics, from which the following example comes.

Example 2 (Grade 8) How much nutrition needs a cyclist on a tour de France?
Unfolding: In this complex problem many aspects play a role: the mountain climbing where potential energy is involved, the efficiency of the human body as an "engine" (about 25%-30%), the velocity (or power) a cyclist can sustain. the air resistance and the efficincy of the bicycle. Besides these physical aspects it can be asked which sorts of drinking and food are appropriate, which leads to interdisciplinary questions.
Reasoning and reduacion: It has to be discussed which of all these parameters are important and which could be safely neglected. Hence the students have to calculate or estimate the different energy consuming parts (e.g. the potential energy of climbing) and to gather information , e.g. about air resistance during cycling and its dependence on the velocity and the rolling resistance.
Evaluation: The results will be compared to each other and to the real need of a cyclist. Then it has to be discussed which parameters have the biggest influence on the energy need of a cyclist and why. Further derivations can be made for similar situations in daily life.

The process of modeling needs support from different sources. One important tool is the use of graphs and diagrams in many variations. (Wells 1995)

2. Diagrams and physics

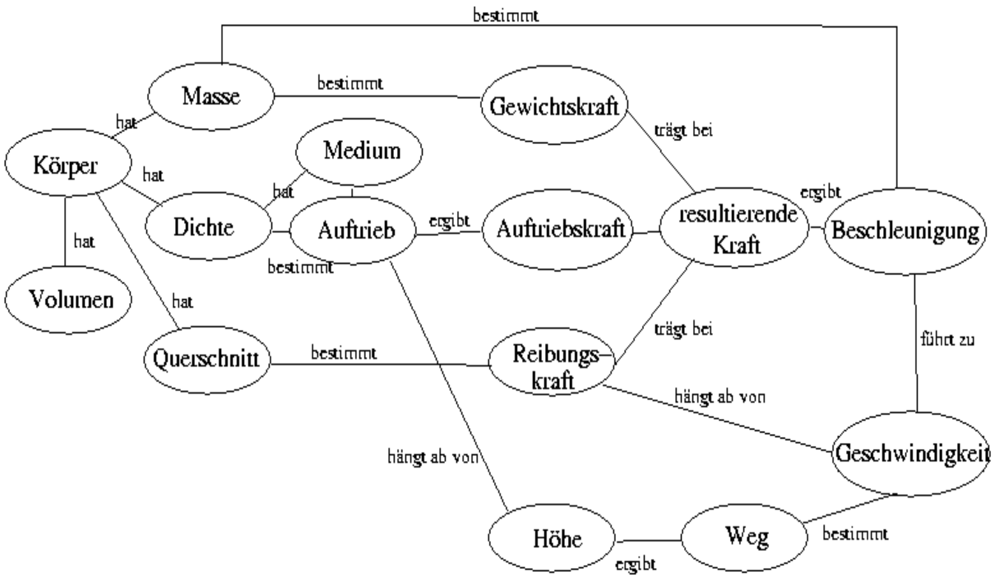
The use of mathematics in a broad sense starts with taking values in experiments, goes to drawing diagrams, interpreting them and in the end deriving a mathematical formulation and interpreting it again. This chain of activitites is the first step towards mathematical modeling.
In general, graphical representations are easy accessible for students. They train them already early in school, beginning in primary school on an elementary basis. In grade 6 they should be able to put values into a coordinate system. The crucial point is the interpretation of the graphical representation in physics connection. Herewith several points are to be considered:

- Description with language requires more knowledge about the interdependencies than simply stating a formula.
- Drawing a graph requires to identify the independent and dependent variables.

Use of both types of representation together are the ingrediences for grasping the structure of content. Diagrams visualize dependencies and as an iconic representation of (experimental) results use a second way to memory besides the formulation by language or mathmatics.
So it is most important to stress the interpretation of a diagram and formulate its meaning in the students' own words.

3. From a concept map to a mathematical model

At school often only very simplified models are treated, for instance in neglecting friction or other parameters. These often do not mirror the experiences of students. The availability of modern computers and simulation software, however, enables students to develop and analyse more complex models that could not be treated analytically. Nevertheless, students need experience in changing between the "world of objects" and the "world of formulas". A concept map could be of help. In the first step the students have to clarify the structure of the discussed topic, be it for instance electricity, mechanics or atomic physics. This requires that students possess a stable knowledge and have insight into the structure of the subject which can be worked out with aid of a concept map. It has shown of advance if the students work together in pairs: there is opportunity to discuss the content and clarify it in this way. The sorting out and structuring can happen using common language and physics terms. Examining the relations then results in establishing the mathematical formulation which has to be translated into formulas and perhaps an algorithm. Below the example of free fall with friction is indicated.



In the following I will describe a proposal for developing the competence in mathematical modeling in more detail (see also Hestenes 1992, Wells 1995).

Proposal

- A problem is to be posed according to the knowledge of the students taking into account the question of motivation (situated learning, context oriented, everyday relevance, fascinating question)
- The students enter into qualitative reasoning with hypotheses and making explicit the theoretical backgraound: What will happen, why will this happen?
- They enter physical reasoning: which effects play a role, which parameters are important, which could or should be neglected, because they only have a small influence or otherwise the problem would be to complicated.
- After this idealisation process a discussion is necessary: how precise can the result be? How can the expected result be tested by further information or an experiment?
  The physical process has to be translated with aid of laws into mathematics in a wider sense by drawing a diagram or deriving a formula.
- Only in the last step by computing, inserting numbers and calculation a testable result is derived. This has again to be interpreted by discussing special cases, errors or deviations from actual values. The limits of the model are analysed.

Mathematics and physics at school

In most curricula it is expected that the students are able to solve problems and to apply physics knowledge to everyday phenomena. Besides making hypotheses, planning and doing experiments this comprises the adequate use of mathematical methods. In a similar sense as explained above also in school mathematics is regarded as a tool for physics. But students have to cope with differences in goals and perspectives between mathematics and physics which causes additional difficulties.
Before discussing possible steps to be taken for bridging the gap between physics and mathematics I will analyse the necessary mathematical tools at school. In the case of mathematics and physics there are some pitfalls because of different semantics that often are not properly addressed:

Differences in terms:

Mathematics	*Physics*
Numbers	Numbers with units
Fraction	Relation
Function in an abstract sense	Functional dependencies
Geometrical objects	Symbolic representations of physical units
Differentiation	Rate of change
Integration	Integration

Mathematical tools at school

A basic understanding of certain features is substantial for physics at school:

- *Functions*: They are most important because they are used for describing dependencies between physical quantities. Examples are the accelerated motion or the radioactive decay.
- *Integration*: This tool is – at least in its most simple form – used for modeling the accumulation of physical quantitites. The most used example is work as the effect of force along a line, summing up along a distance or the derivation of the law for accelerated motion by summing up in time.
- *Differentiation*: One of the most important features in modeling consists in describing the rate of change of a physical quantity in time or space. The earliest example pupils encounter is the velocity.
- *Geometry / vectors*: Geometrical properties are often needed for analysing the motions of objects in space, especially in adding forces or velocities, superposition of motions planetary motions and optics, e.g. theorem of intersecting lines, too.
- *Algebra.* Students need algebraic abilities for manipulating formulas on different levels and in different contexts.

Often the problem is mentioned that the pupils do not yet know the appropriate mathematical tools for treating the physics contents. Partly this may be a problem of synchoronizing the lessons in mathematics and physics, partly it is a principal problem because of the different goals and views in those two subjects.

But with some compromise both subjects may help each other. I give some examples from the saxonian curriculum in Germany.This table shows that although the term of “function” in the mathematics content is not treated before grade 8, proportionalities are already used in grade 6. Proportionalities are examples of linear functions that can be taught without fully referring to the mathematical definition of a function. But nevertheless the graphical representation of a proportionality is a very important tool in recognizing linear dependencies as for instance in Ohm's law or the time-distance law of uniform motion. (Nearly) all functions can be represented graphically which has two advantages:

- The graphics build the bridge between the experimentally measured values and the abstract formula.
- The graphics serve as a visualization, that means an additional help for memory

Graphics serve as a representation of the qualitative behaviour of physical objects.

Grade	*Mathematics*	*Physics*
6	proportionalities	Velocity, density
7	rational numbers	Ohm's Law
8	different types of functions	Meaning of parameters
9	quadratic functions	Laws of accelerated motion
10	trigonometric functions	Periodic motions, oscillations

Conclusion

Developing competencies in scientific thinking is the main task of physics education. The knowledge of mathematical tools is an important part. To achieve that students are able to model phenomena from daily life they need training with suitable problems. Therewith the balance has to be found between the necessary guidance and the students' self directed learning and inquiring. The guidance consists in giving the students a direction how to proceed, which steps to take; the free inquiry requires that the students learn to structure their knowledge and to findsuitable idealizations by themselves. So a task of physics education research is the development of suitable problems and learning environments in order to engage the students in appropriate activities. The connection of phenomena and formulae should be trained more explicitly.

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