Use, Abuse, and Unjustified Neglect of the Action Principle

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Abstract
Traditionally, differential equations dominated physics education; the action principle was used primarily to derive differential equations such as Lagrange's equations. Now the computer allows the action principle to be applied directly from first principles, often bypassing analytic solutions entirely. The action principle can illuminate and unify physics education and physics research from quantum field theory to cosmology.

Computer solutions without solving equations

In classical mechanics it is not true that action along a particle worldline is always minimum. Often it is a minimum, and it is never a maximum. But sometimes action can be a saddle point, which means that, compared with the true worldline, some adjacent curves have a greater value of the action, while some have a smaller value of the action. Seems complicated. However, it is always true that the action along a sufficiently short worldline -- or a sufficiently short segment of a worldline -- is always a minimum. Therefore we can correctly say:

> *A particle follows a worldline such that the action along every small segment is a minimum compared with that along every nearby segment with the same endpoints.*

Looking at this universal principle tells you immediately that computers are perfect for applying action. Computers do increments beautifully. If a computer finds, by whatever means, a worldline along every segment of which action is a minimum, then that is a true worldline, one that a particle can follow.

Figure 1 shows a frame of an interactive program by Slavomir Tuleja (2005). The goal is to transfer a probe in an unpowered trajectory from a parking orbit around earth to a parking orbit around the moon. We use Hamilton's action *S* because the fixed initial and final points must be events; the probe must get to the location of the moon *when the moon is there*. The dots along the trajectory are ticks on the clock carried by the probe, so the representation completely determines the worldline: position vs time. The operator can drag the clock ticks back and forth one at a time to minimize the total action, can add many intermediate ticks to increase the accuracy, and can ask the computer to minimize the action automatically, which it does in a split second. Notice that this analysis moves directly from the action principle to a visualized solution with no

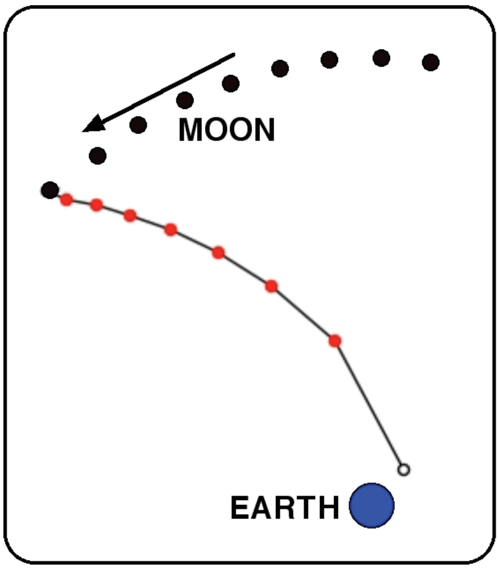

Figure 1. Transfer trajectory between parking orbits around earth and moon. Dots on the trajectory are the events of clock ticks that completely determine the worldline. The operator can add intermediate clock ticks, drag individual ticks to minimize the action, or have the computer minimize action automatically.

intermediate mathematical analysis. And automatically generated spread sheet data, the time and location of every clock tick, can be analyzed to any desired level of detail.

A critic might object that the solution is now too easy; all the student does is push a few buttons. On the contrary, the student has now been freed to investigate a hugely expanded world of possible problems. For example, by changing the time lapse between initial and final events, the student can search for the worldline that minimizes the total rocket impulse required for the transfer from earth orbit to moon orbit. She can try the same for different parking orbits around earth and moon. She can apply a similar program to transfers between earth and mars. The software offers analysis of motion in other potentials as well.

The action principle for special and general relativity and the cosmos

When I was editor of the *American Journal of Physics*, I despaired about the twin paradox, which seemed to poison the literature. Every month some engineer or retired doctor would submit a paper disproving the obviously ridiculous predictions of the twin paradox, thus invalidating special relativity.

In fact the twin paradox is central to the use of action to describe high speed motion: In flat spacetime a true particle worldline yields the longest proper time (wristwatch time, aging) between fixed initial and final events. A general expression for the action uses the Lagrangian *L*, which for low speeds is the difference between the kinetic and potential energies.

$$S \equiv \int L dt$$

A particle moving at any speed in an electromagnetic field has the Lagrangian

$$L = \frac{1}{2} m \mathbf{v} \bullet \mathbf{v} - \frac{q}{c} \mathbf{v} \bullet \mathbf{A}$$

where ϕ is the scalar potential and **A** is the vector potential. Two equations determine all possible worldlines under electromagnetic influence in flat spacetime!

Now, general relativity is weird, but has the following gorgeous simplification: When there are no singularities or gravitational waves, then at every event on a particle worldline you can always find a local inertial frame in which special relativity holds. Since the worldline is the sum of segments in these local frames, and since proper time is an invariant, the same for all observers, therefore in general relativity the worldline of a particle between fixed end events is the one with maximum aging along each small segment. We call this result *the principle of maximal aging*. For low speeds and small curvature of spacetime, the principle of maximal aging reduces to the principle of least action.

How do we find the value $d\tau$ of the proper time (aging) along a segment of a worldline in curved spacetime? From the *metric*, the solution to the field equations. On the left of the metric equation is the increment $d\tau$ of proper time (wristwatch time, aging) between a nearby pair of events on the worldline. On the right side of the metric equation are the corresponding increments of the (arbitrarily chosen) coordinates between that pair of events.

Now, the metric is expressed in increments; manipulating the metric requires only calculus. This means that if, instead of starting with the field equation, we start with the metric solutions, we can introduce general relativity to sophomores using only calculus and the principle of maximal aging. John Archibald Wheeler and I did this in our text *Exploring Black Holes, Introduction to General Relativity*, Fig. 2 (Taylor & Wheeler 2000)

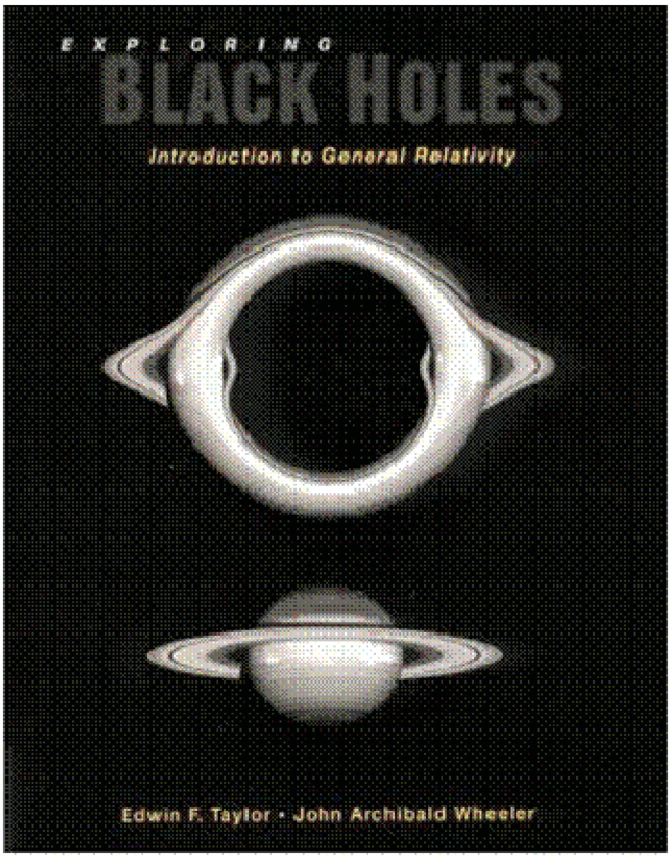

Figure 2. Text Exploring Black Holes by Edwin F. Taylor and John Archibald Wheeler that introduces general relativity to sophomores using the metric and the principle of maximal aging.

The action principle not only tracks particles in curved spacetime. It is also at the root of Einstein's field equations themselves. Hilbert derived the field equations from an action principle, some say before Einstein completed his theory. Landau and Lifshitz do the same for advanced physics students. Thus one can say that action describes the fundamental non-quantum laws of the cosmos.

Quantum mechanics from the bottom

Recall Feynman's formulation of nonrelativistic quantum mechanics, exemplified by the electron:

- The electron explores all possible worldlines between fixed end events that we choose.
- Along each trial worldline the total rotation of the quantum phasor is $S/h = (Action)/h$.
- At the end event, add up phasors for all worldlines to give the resultant quantum amplitude.
- The probability of detecting the electron at the final event is proportional to the squared magnitude of the resultant quantum amplitude.

This is not a new idea! Sixty four years ago it was summarized in the introduction to Feynman's 1942 Ph.D. thesis under John Wheeler: *A* *generalization of quantum mechanics is given in which the central mathematical concept is the analogue of the action in classical mechanics . . . It is only required that some form of least action principle be available . . . if a Lagrangian exists . . . the generalization reduces to the usual form of quantum mechanics. In the classical limit, the quantum equations go over into the corresponding classical ones, with the same action function.* (Brown 2003)

Feynman was a co-recipient of a Nobel Prize for expanding these ideas to quantum field theory, which one can take to be the most fundamental current theory of the very small (a status that string theory has not yet achieved). And quantum field theories can be derived from action. A quote from the Web: "Of all possible fields with a given boundary condition the one that provides an extremum . . . of the action is The Solution."

Thus action carries us seamlessly from the smallest that we know to the largest, the universe as a whole.

Variational principles: parents of action

. . . once the laws of physical theory are expressed as differential equations, the possibility of their reduction to a variational principle is evident from purely mathematical reasoning . . .
(Yourgrau & Mandelstam 1968)

Traditionally differential equations have been the analytic tool of choice in physics, both for education and research. As we have seen, the computer can apply variational principles directly, often eliminating intermediate mathematical analysis. The use of variational principles in current physics text is, at best, spotty. A few scattered examples:

Landau and Lifschitz develop the first two Maxwell Equations from experiment and hand-waving. The last two equations they derive from a variational principle, which they call action.

A standard method for finding the ground state wave function of an atom is to minimize the electromagnetic energy.

The relaxation method is a powerful one for determining the electrostatic field resulting from an array of fixed charges.

Textbooks often miss powerful and simplifying applications of variational principles. For example, Van Baak (1999) replaces Kirchoff's circuit theorems by requirements that (1) current is conserved, and (2) the rate of dissipation of energy is minimized. By using this method, he says, the usual "extravagance of equations is wholly avoided."

Strategies for introducing action

Variational principles -- and action -- are tools, like differential equations, and not fundamental physical principles (though they are related to conservation laws through Noether's theorem; see e.g. Hanc et al. 2004). Still, they are unifying tools that the computer can implement throughout physics education and research. Here are some notes on strategies for introducing them to physics instruction:

1. Sneak bits of action and variational principles into secondary classes and introductory undergraduate courses (see particular examples in Hanc et al. 2003 or Hanc & Taylor 2004).

2. Do NOT make action the primary tool in introductory physics; it is not concrete enough. Introductory students need to feel the pushes and pulls of forces.

3. Use action and variational principles as unifying tools in the remainder of undergraduate and graduate programs.

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Software and background publications are available at www.eftaylor.com/leastaction.html;