Notes for a Modeling Theory of Science, Cognition and Instruction

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Abstract
Modeling Theory provides common ground for interdisciplinary research in science education and the many branches of cognitive science, with implications for scientific practice, instructional design, and connections between science, mathematics and common sense.

I. Introduction

To the grand philosophical question: "*What is man*?"
Aristotle answered:
"*Man is a rational animal*."
Modeling Theory offers a new answer:
"*Man is a modeling animal*!"
Homo modelus!

During the last two decades *Physics Education Research* (PER) has emerged as a viable sub-discipline of physics, with faculty in physics departments specializing in research on learning and teaching physics. There is still plenty of resistance to PER from hard-nosed physicists who are suspicious of any research that smacks of education, psychology or philosophy. However, that is countered by a growing body of results documenting deficiencies in traditional physics instruction and significant improvements with PER-based pedagogy. Overall, PER supports the general conclusion that *science content cannot be separated from pedagogy* in the design of effective science instruction. Student learning depends as much on structure and organization of subject matter as on the mode of student engagement. For this reason, science education research *must* be located in science departments and not consigned to colleges of education.

As one of the players in PER from its beginning, my main concern has been to establish a scientific theory of instruction to guide research and practice. Drawing on my own experience as a research scientist, I identified construction and use of conceptual models as central to scientific research and practice, so I adopted it as the thematic core for a MODELING THEORY of science instruction. From the beginning, it was clear that Modeling Theory had to address cognition and learning in everyday life as well as in science, so it required development of a model-based epistemology and philosophy of science. Thus began a theory-driven MODELING RESEARCH PROGRAM: Applying the theory to design curriculum and instruction, evaluating results, and revising theory and teaching methods accordingly. Fig. 1 provides an overview of the program.

Section II reviews evolution of the Modeling Research Program. Concurrent evolution of Cognitive Science is outlined in Section III. Then comes the main purpose of this paper: To lay foundations for a common modeling theory in cognitive science and science education to drive symbiotic research in both fields. Specific research in both fields is then directed toward a unified account of cognition in common sense, science and mathematics. This opens enormous opportunities for science education research that I hope some readers will be induced to pursue.

Of course, I am not alone in recognizing the importance of *models and modeling* in science, cognition, and instruction. Since this theme cuts across the whole of science, I have surely overlooked many important insights. I can only hope that this paper contributes to a broader dialog if not to common research objectives.

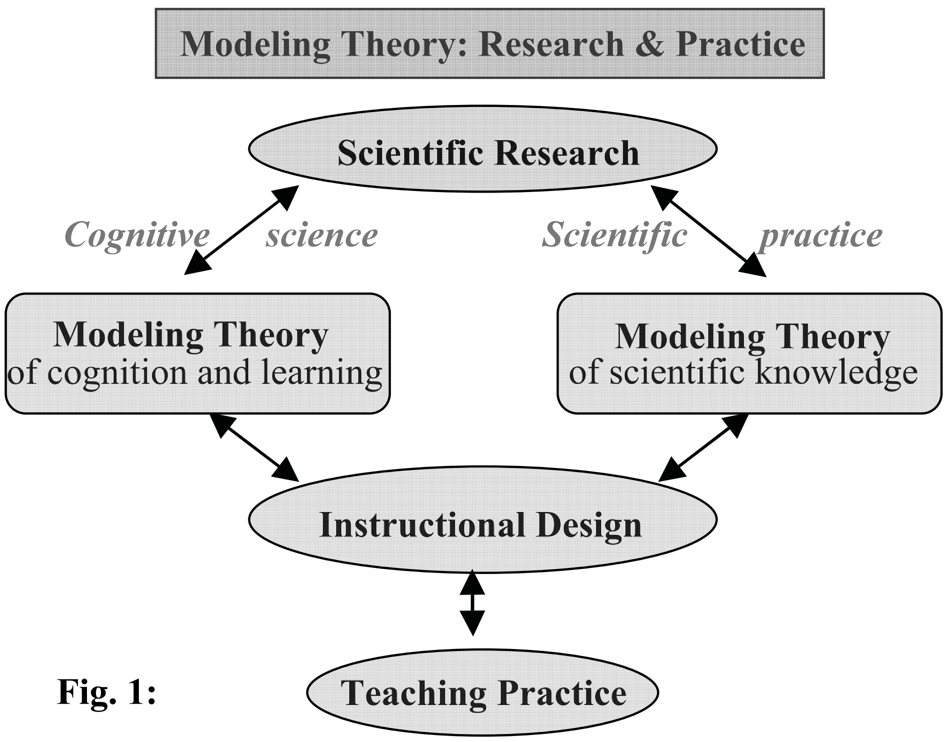


Fig. 1:

II. Evolution of Modeling Instruction

My abiding interest in questions about cognition and epistemology in science and mathematics was initiated by undergraduate studies in philosophy. In 1956 I switched to graduate studies in physics with the hope of finding some answers. By 1976 I had established a productive research program in theoretical physics and mathematics, which, I am pleased to say, is still flourishing today [1]. About that time, activities of my colleagues Richard Stoner, Bill Tillery and Anton Lawson provoked my interest in problems of student learning. The result was my first article advocating Physics Education Research [2].

I was soon forced to follow my own advice by the responsibility of directing PER doctoral dissertations for two outstanding graduate students, Ibrahim Halloun and Malcolm Wells. Halloun started about a year before Wells. In my interaction with them, two major research themes emerged: First, effects on student learning of organizing instruction about models and modeling; Second, effects of instruction on student preconceptions about physics.

The Modeling Instruction theme came easy. I was already convinced of the central role of modeling in physics research, and I had nearly completed an advanced monograph-textbook on classical mechanics with a modeling emphasis [12]. So, with Halloun as helpful teaching assistant, I conducted several years of experiments with modeling in my introductory physics courses.

The second theme was more problematic. I was led to focus on modes of student thinking by numerous discussions with Richard Stoner about results from exams in his introductory physics course. His exam questions called for qualitative answers only, because he believed that is a better indicator of physics understanding than quantitative problem solving. However, despite his heroic efforts to improve every aspect of his course, from the design of labs and problem solving activities to personal interaction with students, class average scores on his exams remained consistently below 40%. In our lengthy discussions of student responses to his questions, I was struck by what they revealed about student thinking and its divergence from the physics he was trying to teach. So I resolved to design a test to evaluate the discrepancy systematically. During the next several years I encountered numerous hints in the literature on what to include. When Halloun arrived, I turned the project over to him to complete the hard work of designing test items, validating the test and analyzing test results from a large body of students.

The results [3, 4] were a stunning surprise! surprising even me! so stunning that the journal editor accelerated publication! With subsequent improvements [5], the test is now known as the *Force Concept Inventory* (FCI), but that has only consolidated and enhanced the initial results. Instructional implications are discussed below in connection with recent developments. For the moment, it suffices to know that the FCI was immediately recognized as a reliable instrument for evaluating the effectiveness of introductory physics instruction in both high school and college.

Five major papers [6-10] have been published on Modeling Theory and its application to instruction. These papers provide the theoretical backbone for the *Modeling Instruction Project* [11], which is arguably the most successful program for high school physics reform in the U.S. if not the world. Since the papers have been seldom noted outside that project, a few words about what they offer is in order.

The first paper [6] provides the initial theoretical foundation for Modeling Theory and its relation to cognitive science. As *modeling* has become a popular theme in science education in recent years, it may be hard to understand the resistance it met in 1985 when my paper was first submitted. Publication was delayed for two years by vehement objections of a referee who was finally overruled by the editor. Subsequently, the paper was dismissed as mere speculation by empiricists in the PER community, despite the fact that it was accompanied by a paper documenting successful application to instruction.

Nevertheless, this paper provided the initial conceptual framework for all subsequent developments in modeling instruction. It must be admitted, though, the paper is a difficult read, more appropriate for researchers than teachers.

Paper [7] is my personal favorite in the lot, because it exorcises the accumulated positivist contamination of Newtonian physics in favor of a model-centered cognitive account. For the first time it breaks with tradition to formulate all six of Newton's laws. This is important pedagogically, because all six laws were needed for complete coverage of the "Force concept" in designing the FCI [5]. Moreover, explicit formulation of the Zeroth Law (about space and time) should interest all physicists, because that is the part of Newtonian physics that was changed by relativity theory. Beyond that, the paper shows that Newton consciously employed basic modeling techniques with great skill and insight. Indeed, Newton can be credited with formulating the first set of rules for MODELING GAMES that scientists have been playing ever since.

Paper [11] applies Modeling Theory to instructional design, especially the design of software to facilitate modeling activities. Unfortunately, the R&D necessary to build such software is very expensive, and funding sources are still not geared to support it.

In contrast to the preceding theoretical emphasis, papers [8, 9] are aimed at practicing teachers. Paper [8] describes the results of Wells' doctoral thesis, along with instructional design that he and I worked out together and his brilliant innovations in modeling discourse management. His invention of the portable *whiteboard* to organize student discourse is propagating to classrooms throughout the world. Sadly, terminal illness prevented him from contributing to this account of his work.

Wells' doctoral research deserves recognition as one of the most successful and significant pedagogical experiments ever conducted. He came to me as an accomplished teacher with 30 years experience who had explored every available teaching resource. He had already created a complete system of activities to support student-centered inquiry that fulfills every recommendation of the *National Science Education Standards* today. Still he was unsatisfied. Stunned by the performance of his students on the FCI-precursor, he resolved to adapt to high school the ideas of modeling instruction that Halloun and I were experimenting with in college. The controls for his experiment were exceptional. As one control, he had complete data on performance of his own students without modeling. Classroom activities for treatment and control groups were identical. The only difference was that discourse and activities were focused on models with emphasis on eliciting and evaluating the students' own ideas. As a second control, posttest results for the treatment group were compared to a well-matched group taught by traditional methods over the same time period. The comparative performance gains of his students were unprecedented. However, I am absolutely confident of their validity, because they have been duplicated many times, not only by Wells but others that followed.

I was so impressed with Wells' results that I obtained in 1989 a grant from the U.S. *National Science Foundation*, to help him develop *Modeling*

Workshops to inspire and enable other teachers to duplicate his feat. Thus began the *Modeling Instruction Project*, which, with continuous NSF support, has evolved through several stages with progressively broader implications for science education reform throughout the United States. Details are available at the project website [11]. None of this, including my own involvement, would have happened without the pioneering influence of Malcolm Wells.

III. Evolution of Cognitive Science

Cognitive science grew up in parallel with PER and Modeling Theory. With the aim of connecting the strands, let me describe the emergence of cognitive science from the perspective of one who has followed these developments from the beginning. Of course, the mysteries of the human thought have been the subject of philosophical contemplation since ancient times, but sufficient empirical and theoretical resources to support a genuine science of mind have been assembled only recently. Box 1 outlines the main points I want to make.

I regard the *formalist* movement in mathematics as an essential component in the evolution of *mathematics as the science of structure*, which is a central theme in our formulation of Modeling Theory below. Axioms are often dismissed as mathematical niceties, inessential to science. But it should be recognized that axioms are essential to Euclidean geometry, and without geometry there is no science. I believe that the central figure in the formalist movement, David Hilbert, was the first to recognize that axioms are actually definitions! ***Axioms define the structure in a mathematical system, and structure makes rational inference possible***!

Equally important to science is the operational structure of scientific measurement, for this is essential to relate theoretical structures to experiential structures in the physical world. This point has been made most emphatically by physicist Percy Bridgeman, with his concept of *operational definitions* for physical quantities (but see [7] for qualifications). However, to my mind, the deepest analysis of scientific measurement has been made by Henri Poincaré, who explained how measurement conventions profoundly influence theoretical conceptions. In particular, he claimed that curvature of physical space is not a fact of nature independent of how measurements are defined. This claim has long been inconclusively debated in philosophical circles, but recently it received spectacular confirmation [14].

Following a long tradition in rationalist philosophy, the formalist movement in mathematics and logic has been widely construed as the foundation for a theory of mind, especially in Anglo-American analytic philosophy. This is an egregious mistake that has been roundly criticized by George Lakoff and Mark Johnson [17-21] in the light of recent developments in cognitive science. Even so, as already suggested, formalist notions play an important role in characterizing structure in cognition.

The creation of serial computers can be construed as technological implementation of operational structures developed in the formalist tradition. It soon stimulated the creation of *information processing psychology*, with the notion that cognition is all about symbol processing. I was right up to date in applying this egregious mistake to physics teaching [2]. Even so, most of the important research results and insights that I reported survive reinterpretation when the confusion between cognition and symbol processing is straightened out. Symbol processing is still a central idea in computer science and Artificial Intelligence (AI), but only the ill-informed confuse it with cognitive processes.

> **Box 1: Emergence of Cognitive Science**
> I. Scientific Precursors
> • **<u>Formalist</u> mathematics and logic** (~1850-1940)
> – axioms & standards for rigorous proof
> – reasoning by rules and algorithms
> • **<u>Operationalism</u>** (Bridgeman, 1930)
> • **<u>Conventionalism</u>** (Poincaré, 1902)
> • Gestalt psychology (~1915-1940)
> • Genetic Epistemology (Piaget, ~1930-1960)
> **II. Emergence of computers and computer science**
> (~1945-1970) implementing operational structures
> **III. First Generation Cognitive Science** (~1960-1980)
> • "Brain is a *serial computer*" metaphor
> • "Mind is a computer software system"
> • Information processing psychology & AI
> – **<u>Thinking is symbol manipulation</u>**
> – functionalism (details about the brain irrelevant)
> **IV. Second Generation Cognitive Science** (~1983--)
> • <u>Neural network level</u>
> – Brain is a *massively parallel* dynamical system
> – **<u>Thinking is pattern processing</u>**
> • **Cognitive phenomenology** at the <u>functional level</u>:
> – empirical evidence for mental modeling
> is accumulating rapidly from many sources.

I tried to link the dates in Box 1 to significant events in each category. I selected the date 1983 for the onset of second generation cognitive science, because I had the privilege of co-organizing the very first conference devoted exclusively to what is now known as *cognitive neuroscience.* It still took several years to overcome the heavy empiricist bias of the neuroscience community and establish neural network modeling as a respectable activity in the field. The consequence has been a revolution in thinking about thinking that we aim to exploit.

IV. Modeling Research in Cognitive Science

With its promise for a universal science of mind, research in cognitive science cuts across every scientific discipline and beyond. Box 2 lists research that I see as highly relevant to the Modeling Theory I am promoting. The list is illustrative only, as many of my favorites are omitted. These scientists are so productive that it is impractical to cite even their most important work. Instead, I call attention to the various research themes, which will be expanded with citations when specifics are discussed.

References [15, 16] provide an entrée to the important work of Giere, Nercessian and Gentner, which has so much in common with my own thinking that it may be hard to believe it developed independently. This illustrates the fact that significant ideas are implicit in the culture of science waiting for investigators to explicate and cultivate as their own.

In sections to follow, I emphasize alignment of Modeling Theory with Cognitive Linguistics, especially as expounded by George Lakoff [17-20]. Language is a window to the mind, and linguistic research has distilled a vast corpus of data to deep insights into structure and use of language. My objective is to apply these insights to understanding cognition in science and mathematics. Cognitive Linguistics makes this possible, because it is a reconstruction of linguistic theory aligned with the recent revolution in Cognitive Science.

V. Constraints from Cognitive Neuroscience

Cognitive neuroscience is concerned with explaining cognition as a function of the brain. It bridges the interface between psychology and biology. The problem is to match cognitive theory at the psychological level with neural network theory at the biological level. Already there is considerable evidence supporting the *working hypothesis* that ***cognition*** (at the psychological level) ***is grounded in the sensory-motor system*** (at the biological level).
The evidence is of three kinds:

- <u>*Soft constraints*</u>*:* Validated *models of cognitive structure* from cognitive science, especially cognitive linguistics.
- <u>*Hard constraints*</u>*:* Identification of *specific neural architectures and mechanisms* sufficient to support cognition and memory.
- <u>*Evolutionary constraint*</u>: A plausible account of how the brain could have evolved to support cognition.

A few comments will help fix some of the issues.

Box 2: Modeling Research in Cognitive Science

Philosophy of Science
Ronald Giere (Model-based philosophy of science)
Jon Barwise (Deductive inference from diagrams)

History and Sociology of Science
Thomas Kuhn (Research driven by Exemplars)
Nancy Nercessian (Maxwell's analogical modeling)

Cognitive Psychology
Dedre Gentner (Analogical reasoning)
Philip Johnson-Laird (Inference with mental models)
Barbara Tversky (Spatial mental models vs. visual imagery)

Cognitive Linguistics
George Lakoff (Metaphors & radial categories)
Ronald Langacker (Cognitive grammar & image schemas)

Cognitive Neuroscience
Michael O'Keeffe (Hippocampus as a Cognitive Map)
Stephen Grossberg (Neural network theory)

Physics Education Research
Andy diSessa (Phenomenological primitives)
John Clement (Bridging analogies)
Information & Design Sciences
UML: Universal Modeling Language
& Object-Oriented Programming

Biology tells us that brains evolved adaptively to enable navigation to find food and respond to threats. Perception and action are surely grounded in identifiable brain structures of the sensory-motor system. However, no comparable brain structures specialized for cognition have been identified. This strongly suggests that cognition too is grounded in the sensory-motor system. The main question is then: what adaptations and extensions of the sensory-motor system are necessary to support cognition?

I hold that introspection, despite its bad scientific reputation, is a crucial source of information about cognition that has been systematically explored by philosophers, linguists and mathematicians for ages. As Kant was first to realize and Lakoff has recently elaborated [20], the very structure of mathematics is shaped by hard constraints on the way we think. A major conclusion is that geometric concepts (grounded in the sensory-motor system) are the prime source of relational structures in mathematical systems.

I am in general agreement with Mark Johnson's NeoKantian account of cognition [21], which draws on soft constraints from Cognitive Linguistics. But it needs support by reconciliation with hard constraints from sensory-motor neuroscience. That defines a promising direction for research in Cognitive Neuroscience. Let me reiterate my firm opinion [6] that the research program of Stephen Grossberg provides the best theoretical resources to pursue it.

VI. System, Model & Theory; Structure & Morphism

The terms 'system' and 'model' have been ubiquitous in science and engineering since the middle of the twentieth century. Mostly these terms are used informally, so their meanings are quite variable. But for the purposes of Modeling Theory, we need to define them as sharply as possible. Without duplicating my lengthy discussions of this matter before [7-10], let me reiterate some key points with an eye to preparing a deeper connection to cognitive theory in the next section.

> **Box 3:** A conceptual **MODEL** is defined by specifying **five types of structure**:
>
> (a) **systemic structure**:
> - ***composition*** (internal parts (objects) in the system)
> - ***environment*** (external agents linked to the system)
> - ***connections*** (external and internal links)
>
> (b) **geometric structure:**
> - ***position*** with respect to a reference frame (external)
> - ***configuration*** (geometric relations among the parts)
>
> (c) **object structure**:
> - intrinsic properties of the parts
>
> (d) **interaction structure:**
> - properties of (causal) links
>
> (e) **temporal (event) structure**:
> - temporal change in structure of the system

I define a **SYSTEM** *as a set of related objects.* Systems can be of any kind depending on the kind of object. A system itself is an object, and the objects of which it is composed may be systems. In a *conceptual system* the objects are *concepts*. In a *material system* the objects are material *things*. Unless otherwise indicated, we assume that the systems we are talking about are material systems. A material system can be classified as physical, chemical or biological, depending on relations and properties attributed to the objects.

The **STRUCTURE** of a system is defined as the set of relations among objects in the system. This includes the relation of "belonging to," which specifies COMPOSITION, the set of objects belonging to the system. A universal finding of science is that all material systems have geometric, causal and temporal structure, and no other (metaphysical) properties are needed to account for their behavior. According to Modeling Theory, science comes to know objects in the real world not by direct observation, but by constructing conceptual models to interpret observations and represent the objects in the mind. This epistemological precept is called *Constructive Realism* by philosopher Ronald Giere.

I define a conceptual **MODEL** as a *representation of **structure*** in a material system, which may be real or imaginary. The *possible* types of structure are summarized in Box 3. I have been using this definition of model for a long time, and I am yet to find a model in any branch of science that cannot be expressed in these terms.

Models are of many kinds, depending on their purpose. All models are idealizations, representing only structure that is *relevant* to the purpose, not necessarily including all five types of structure in Box 3. The prototypical kind of model is a map. Its main purpose is to specify geometric structure (relations among places), though it also specifies objects in various locations. Maps can be extended to represent motion of an object by a path on the map. I call such a model a *motion map*. Motion maps should not be confused with *graphs* of motion, though this point is seldom made in physics or math courses. In relativity theory, motion maps and graphs are combined in a single *spacetime map* to represent integrated *spatiotemporal event structure.*

A ***mathematical model*** represents the structure of a system by quantitative variables of two types: *state variables*, specifying composition, geometry and object properties; *interaction variables*, specifying links among the parts and with the environment [6]. A ***process model*** represents temporal structure as change of state variables. There are two types. A *descriptive model* represents change by explicit functions of time. A *dynamical model* specifies equations of change determined by interaction laws. *Interaction laws* express interaction variables as functions of state variables.

A ***scientific THEORY*** is defined by a system of general principles (or **Laws**) specifying a class of state variables, interactions and dynamics (modes of change) [6, 7]. Scientific practice is governed by two kinds of law:

I. *Statutes*: General Laws defining the domain and structure of a Theory
(such as Newton's Laws and Maxwell's equations)

II. *Ordinances*: Specific laws defining models
(such as Galileo's law of falling bodies and Snell's law)

The *content* of a scientific theory is a population of validated models. The statutes of a theory can be validated only indirectly through validation of models.

Laws defining state variables are intimately related to *Principles of Measurement* (also called correspondence rules or operational definitions) for assigning measured values to states of a system. A model is *validated to the degree* that measured values (data) match predicted values determined by the model. The class of systems and range of variables that match a given model is called its *domain of validity*. The domain of validity for a theory is the union of the validity domains for its models.

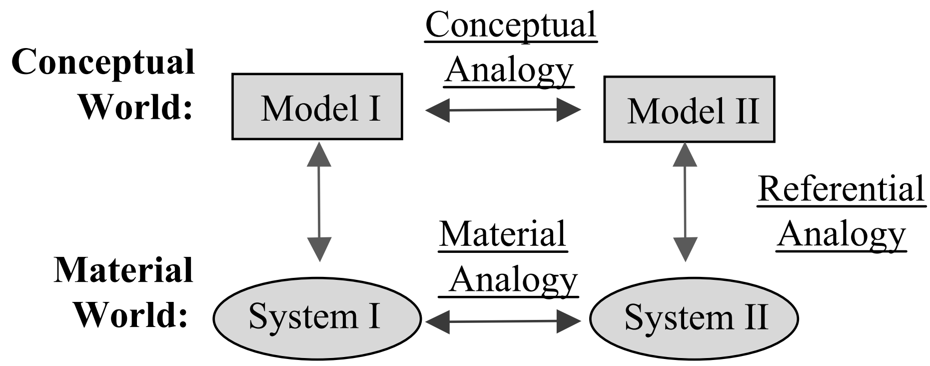


Fig. 3: Three Kinds of Analogy

Empirical observation and measurement determine an *analogy between a given model and its referent* (a system). I call this a *referential analogy*. An ***analogy*** is defined as a *mapping of structure* from one domain (*source*) to another (*target*). The mapping is always partial, which means that some structure is not mapped. (For alternative views on analogy see [16].) Analogy is ubiquitous in science, but often goes unnoticed. Several different kinds are illustrated in Figs. 3&4.

Conceptual analogies between models in different domains are common in science and often play a generative role in research. Maxwell, for example, explicitly exploited electrical– mechanical analogies. An analogy specifies *differences* as well as *similarities* between source and target. For example, similar models of wave propagation for light, sound and water and ropes suppress confounding differences, such as the role of an underlying medium. Such differences are still issues in scientific research as well as points of confusion for students.

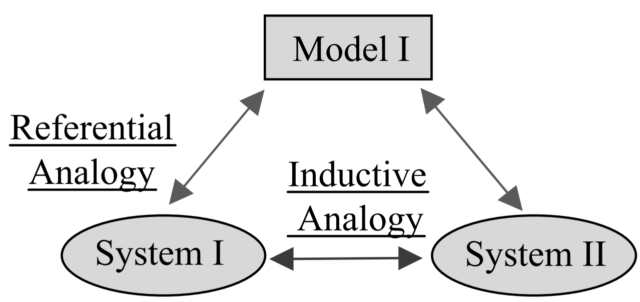


Fig. 4: Material equivalence

A *material analogy* relates structure in different material systems or processes; for example, geometric similarity of a real car to a scale model of the car. An important case that often goes unnoticed, because it is so subtle and commonplace, is *material equivalence* of two material objects or systems, whereby they are judged to be the *same or identical*. I call this an *inductive analogy*, because it amounts to matching the objects to the same model (Fig. 4). I submit that this matching process underlies classical *inductive inference,* wherein repeated events are attributed to a single mechanism.

One other analogy deserves mention, because it plays an increasingly central role in science: the *analogy between conceptual models and **computer models***. The formalization of mathematics has made it possible to imbed every detail in the structure of conceptual models in computer programs, which, running in simulation mode, can emulate the behavior of material systems with stunning accuracy. More and more, computers carry out the empirical function of matching models to data without human intervention. However there is an essential difference between computer models and conceptual models, which we discuss in the next section.

Considering the multiple, essential roles of analogy just described, I recommend formalizing the concept of analogy in science with the technical term **MORPHISM**. In mathematics a *morphism is a structure-preserving mapping*: Thus the terms *homomorphism* (preserves algebraic structure) and *homeomorphism* (preserves topological structure). Alternative notions of analogy are discussed in [16].

The above characterization of science by Modeling Theory bears on deep epistemological questions long debated by philosophers and scientists. For example:

- In what sense can science claim ***objective knowledge*** about the material world?
- To what degree is observed structure inherent in the material world and

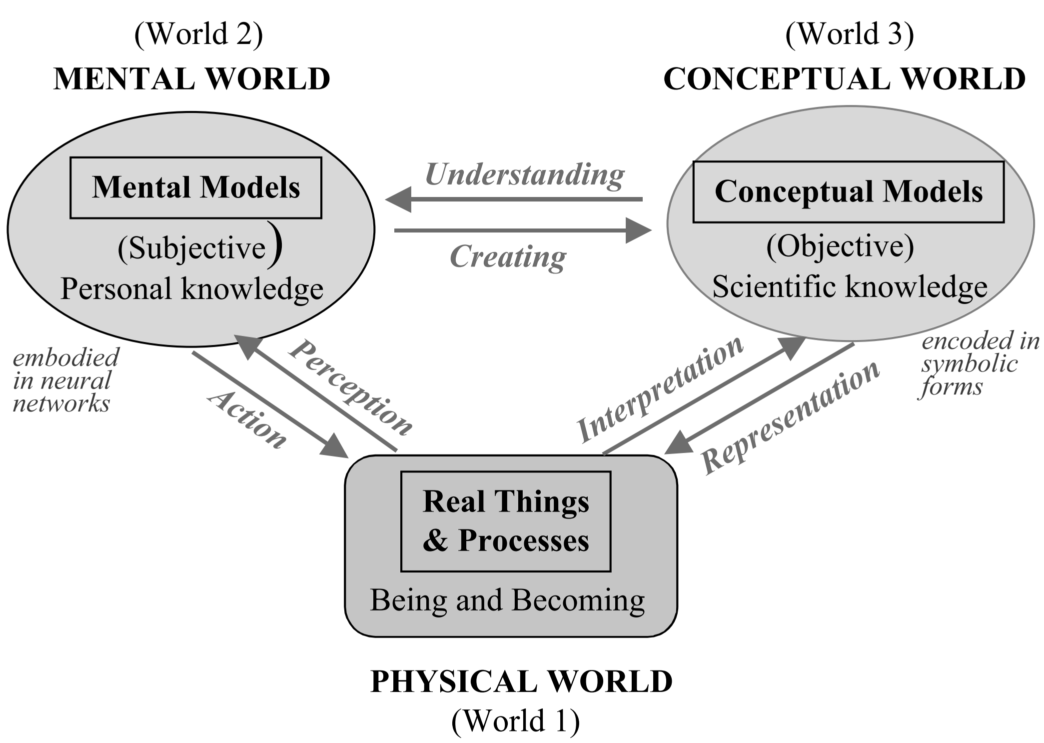


Fig. 5. **Mental models** vs. **Conceptual models**

independent of the observer?
- What determines the **structure categories** for conceptual models in Box 3?

In regard to the last question, I submit in line with Lakoff and Johnson [18, 19, 21] that these are *basic categories of cognition grounded in the human sensory-motor system*. This suggests that answers to all epistemological questions depend on our theory of cognition, to which we now turn.

VII. Modeling Structure of Cognition

If cognition in science is an extension of common sense, then the structure of models in science should reflect structure of cognition in general. To follow up this hint I outline a **Modeling Theory of Cognition**. The theory begins with a crucial distinction between mental models and conceptual models (Fig. 5). **Mental models** are private constructions in the mind of an individual. They can be elevated to **conceptual models** by encoding model structure in symbols that activate the individual's mental model and corresponding mental models in other minds. Just as Modeling Theory characterizes science as construction and use of shared *conceptual models*, I propose to characterize cognition as construction and manipulation of private *mental models*.

As already mentioned, the idea that mental models are central to cognition is commonplace in cognitive science. However, it has yet to crystallize into commonly accepted theory, so I cannot claim that other researchers will approve of the way I construe their results as support for Modeling Theory. The most extensive and coherent body of evidence comes from cognitive linguistics, supporting the **revolutionary thesis**: *Language does not refer directly to the world, but rather to mental models and components thereof! Words serve to activate, elaborate or modify mental models, as in comprehension of a narrative.*

This thesis rejects all previous versions of semantics, which located the referents of language outside the mind, in favor of **cognitive semantics**, which locates referents inside the mind. I see the evidence supporting cognitive semantics as overwhelming [17-24], but it must be admitted that some linguists are not convinced, and many research questions remain.

My aim here is to assimilate insights of cognitive linguistics into Modeling Theory and study implications for cognition in science and mathematics. The first step is to sharpen our definition of concept. Inspired by the notion of '*construction*' in cognitive linguistics [25], I define a ***concept*** *as a {form, meaning} pair represented by a symbol (or symbolic construction)*, as schematized in Fig. 6. The ***meaning*** *is given by a mental model or schema* called a **prototype**, and the ***form*** *is the structure or a substructure of the prototype.*

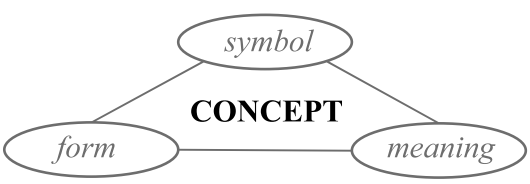


Fig. 6: Concept triad

This is similar to the classical notion that the meaning of a symbol is given by its *intension* and *extension*, but the differences are profound.

For example, the prototype for the concept *right triangle* is a mental image of a triangle, and its form is a system of relations among its constituent vertices and sides. The concept of *hypotenuse* has the same prototype, but its form is a substructure of the triangle. This kind of substructure selection is called **profiling** in cognitive linguistics. Note that different individuals can agree on the meaning and use of a concept even though their mental images may be different. We say that their mental images are *homologous*.

In my definition of a concept, the form is derived from the prototype. Suppose the opposite. I call that a ***formal concept***. That kind of concept is common in science and mathematics. For example the concept of *length* is determined by a system or rules and procedures for measurement that determine the structure of the concept. To understand the concept, each person must embed the structure in a mental model of his own making. Evidently formal concepts can be derived from "informal concepts" by explicating the implicit structure in a prototype. I submit that this process of *explication* plays an important role in both developing and learning mathematics.

Like a percept, a concept is an irreducible whole, with gestalt structure embedded in its prototype. Whereas a percept is activated by sensory input, a concept is activated by symbolic input. Concepts can be combined to make more elaborate concepts, for which I recommend the new term **construct** to indicate that it is composed of irreducible concepts, though its wholeness is typically than the "sum" of its parts.

We can apply the definition of 'concept' to sharpen the notion of 'conceptual model,' which was employed informally in the preceding section. A **conceptual model** *is now defined as a concept (or construct if you will) with the additional stipulation that the structure of its referent be encoded in its representation by a symbolic construction*, or figure, or some other inscription. Like a concept, a conceptual model is characterized by a triad, as depicted in Fig. 7.

To emphasize the main point: the symbols for concepts refer to mental models (or features thereof), which may or may not correspond to actual material objects (as suggested in Fig.7). Though every conceptual model refers to a mental model, the converse is not true. The brain creates all sorts of mental constructions, including mental models, for which there are no words to express. I refer to such constructions as **ideas** or **intuitions**. *Ideas and intuitions are elevated to concepts by creating symbols to represent them!*

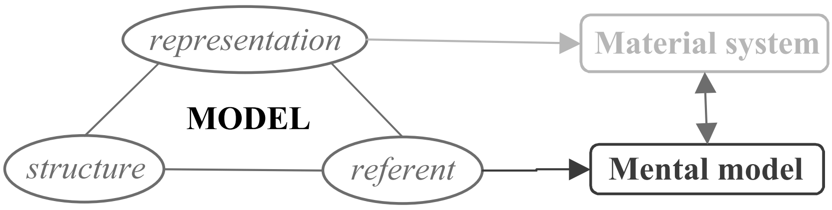


Fig. 7: Conceptual model

My definitions of 'concept' and 'conceptual model' have not seen print before, so others may be able to improve them. But I believe they incorporate the essential ideas. The main task remaining is to elaborate the concept of mental model with reference to empirical support for important claims.

The very idea of *mental model* comes from introspection, so that is a good place to start. However, introspection is a notoriously unreliable guide even to our own thinking, partly because most thinking is unconscious processing by the brain. Consequently, like the tip of an iceberg, only part of a mental model is open to direct inspection. Research has developed means to probe more deeply.

Everyone has imagination, the ability to conjure up an image of a situation from a description or memory. What can that tell us about mental models? Some people report images that are picture-like, similar to actual visual images. However, others deny such experience, and blind people are perfectly capable of imagination. Classical research in this domain found support for the view that *mental imagery is internalized perception,* but not without critics.

Box 4: Spatial MENTAL models

- are ***schematic***, representing only some features,
- are ***structured,*** consisting of *elements and relations*.
- **Elements are typically *objects*** (or reified things).
- **Object *properties* are idealized** (points, lines or paths).
- Object models are always ***placed in a background***
- (context or **frame**).
- Individual objects are ***modeled separately*** from the frame, so they can move around in the frame.

Barbara Tversky and collaborators [26] have tested the classical view by comparison to mental model alternatives. Among other things, they compared individual accounts of a visual scene generated from narrative with accounts generated from direct observation and found that they are *functionally equivalent.* A crucial difference is that perceptions have a fixed point of view, while mental models allow change in point of view. Furthermore, spatial mental models are more schematic and categorical than images, capturing some features of the object but not all and incorporating information about the world that is not purely perceptual. Major characteristics of spatial mental models are summarized in Box 4. The best fit to data is a **spatial framework model**, where each object has an *egocentric frame* consisting of mental extensions of three body axes.

The general conclusion is that ***mental models represent states of the world*** as conceived, not perceived. To know a thing is to form a mental model of it. The details in Box 4 are abundantly supported by other lines of research, especially in cognitive linguistics, to which we now turn.

In the preceding section we saw that concepts of ***structure** and **morphism** provide the foundation for models and modeling practices in science* (and, later

I will claim, for mathematics as well). My purpose here is to link those concepts to the extensive cognitive theory and evidence reviewed by Lakoff and company [17-24], especially to serve as a guide for those who wish to mine the rich lode of insight in this domain. To that end, I have altered Lakoff's terminology somewhat but I hope not misrepresented his message.

I claim that **all reasoning is inference from structure**, so I seek to identify basic cognitive structures and understand how they generate the rich conceptual structures of science and mathematics. The following major themes are involved:

- Basic concepts are irreducible **cognitive primitives** grounded in sensory-motor experience.
- All other conceptual domains are structured by **metaphorical extension** from the basic domain.
- Cognition is organized by **semantic frames**, which provide background structure for distinct conceptual domains and modeling in **mental spaces**.

Only a brief orientation to each theme can be given here.

Metaphors are morphisms in which structure in the source domain is projected into the target domain to provide it with structure. The process begins with **grounding metaphors**, which project structural primitives from basic concepts. A huge catalog of metaphors has been compiled and analyzed to make a strong case that all higher order cognition is structured in this way.

Semantic frames provide an overall conceptual structure linking systems of related concepts (including the words that express them). In mathematics, the frames may be general conceptual systems such as arithmetic and geometry or subsystems thereof. Everyday cognition is structured by a great variety of frames, such as the classic *restaurant frame* that that provides a context for modeling what happens in a restaurant. A semantic frame for a temporal sequence of events, such as *dining* (ordering, eating and paying for a meal), is called a **script**.

Fauconnier has coined the term **mental spaces** for the arenas in which mental modeling occurs [23, 24]. Especially significant is the concept of **blending**, whereby distinct frames are blended to create a new frame. The description of cognitive processes in such terms is in its infancy but very promising.

As cognitive grounding for science and mathematics, we are most interested in **basic concepts** of **space, time and causality**. Their prototypes, usually called *schemas,* provide the primitive structures from which all reasoning is generated. There are two kinds, called *image schemas and aspectual schemas.*

Image schemas provide common structure for spatial concepts and spatial perceptions, thus linking language with spatial perception. The world's languages use a relatively small number of image schemas, but they incorporate spatial concepts in quite different ways — in English mostly with prepositions. Some prepositions, such as *in/out* and *from/to,* express topological concepts, while others, such as *up/down* and *left/right,* express directional concepts.

The schema for each concept is a structured whole or ***gestalt***, where in the parts have no significance except in relation to the whole. For example, the *container schema* (Fig. 8) consists of a boundary that separates interior and exterior spaces. The preposition *in* profiles the interior, while *out* profiles the exterior.

The container schema provides the structure for the general concepts of *containment* and *space* as a container. The alternative notion of space as a set of points (locations) was not invented until the nineteenth century. The contrast between these two concepts of space has generated tension in the foundations of mathematics that is still not resolved to everyone's satisfaction.

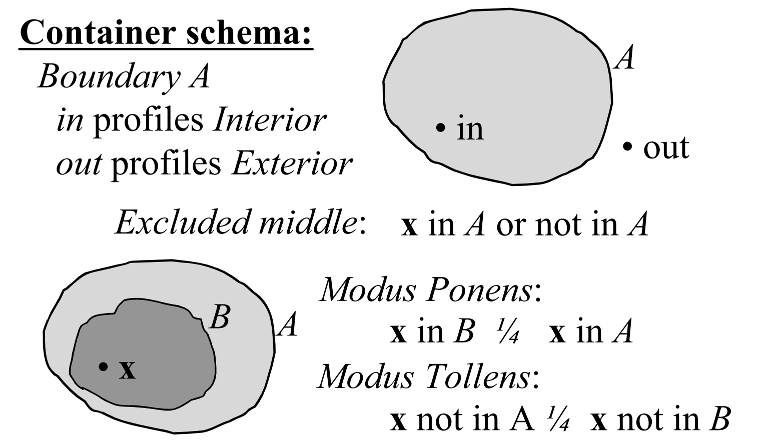


Fig. 8: Container Schema Logic

By metaphorical projection, the container schema structures many conceptual domains. In particular, as Lakoff explains at length, the Categories-are-containers metaphor provides propositional logic with cognitive grounding in the inherent logic of the container schema (illustrated in Fig. 8). More generally, container logic is the logic of *part-whole structure*, which underlies the concepts of set and system (Box 4).

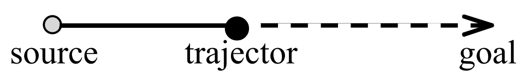


Fig. 9: Source-Path-Goal Schema

Aspectual schemas structure events and actions. The prototypical *aspectual concept* is the *verb*, of which the reader knows many examples. The most fundamental aspectual schema is the basic *schema for motion* (Fig. 9), called the **Source-Path-Goal schema** by linguists, who use *trajector* as the default term for any object moving along a path. This schema has its own logic, and provides cognitive structure for the concepts of *continuity* and *linear order* in mathematics. Indeed, Newton conceived of curves as traced out by moving points, and his First Law of Motion provides grounding for the concept of time on the more basic concept of motion [7]. Indeed, the Greek concept of a curve as a *locus* of points suggests the action of drawing the curve. In physics the concept of motion is integrated with concept of space, and the geometry of motion is called *kinematics*.

Though the path schema of Fig. 9 is classified as aspectual in cognitive linguistics, evidence from cognitive neuroscience and perceptual psychology suggests that it should regarded as an image schema. It is a mistake to think that visual processing is limited to static images. In visual cortex motion is processed concurrently with form. Even young children can trace the path of a thrown ball, and the path is retained mentally as a kind of afterimage, though, like most of visual processing, it remains below the radar of consciousness.

Clearly, the basic concepts of structure and quantity come from geometry. Evidently the *general concept of structure* is derived from geometry by metaphorical projection to practically every conceptual domain. An obvious example is the general concept of *state space,* where states are identified with locations.

Categories are fundamental to human thought, as they enable distinctions between objects and events. One of the pillars of cognitive linguistics is Eleanor Rosch's discovery that ***Natural Categories*** are determined by mental prototypes. This should be contrasted with the classical concept of a ***Formal Category*** for which membership is determined by a set of defining properties, a noteworthy generalization of the container metaphor. The notion of categories as containers cannot account for a mountain of empirical evidence on natural language use.

Natural categories (commonly called ***Radial categories***) are discussed at great length by Lakoff [18], so there is no need for details here. The term "radial" expresses the fact that natural categories have a radial structure of subordinate and superordinate categories with a central category for which membership is determined by matching to a prototype. The matching process accounts for fuzziness in category boundaries and graded category structure with membership determined by partial matching qualified by *hedges*, such as "It looks like a bird, but . . ."

The upshot is that the structure of natural categories is derived from prototypes whereas for formal categories structure is imposed by conventions. As already noted for formal concepts, formal categories play an essential role in creating objective knowledge in science and mathematics. However, the role of radial categories in structuring scientific knowledge has received little notice [27].

Most human reasoning is ***inference from mental models***. We can distinguish several types of **model-based reasoning**:

- **Abductive,** to complete or extend a model, often guided by a semantic frame in which the model is embedded.
- **Deductive**, to extract substructure from a model.
- **Inductive**, to match models to experience.
- **Analogical**, to interpret or compare models.
- **Metaphorical***,* to infuse structure into a model.
- **Synthesis,** to construct a model, perhaps by analogy or blending other models.
- **Analysis,** to profile or elaborate implicit structure in a model.

Justification of model-based reasoning requires translation from mental models to ***inference from conceptual models*** that can be publicly shared, like the scientific models in the preceding section.

In contrast, **formal reasoning** is computational, using axioms, production rules and other procedures. It is the foundation for rigorous proof in mathematics and formal logic. However, I daresay that mathematicians and even logicians reason mostly from mental models. Model-based reasoning is more general and powerful than propositional logic, as it integrates multiple representations of information (propositions, maps, diagrams, equations) into a coherently structured mental model. Rules and procedures are central to the formal concept of inference, but they can be understood as prescriptions for operations on mental models as well as on symbols.

We have seen how Modeling Theory provides a theoretical framework for cognitive science that embraces the findings of cognitive linguistics. Thus it provides the means for scientific answers to long-standing philosophical questions, such as: What is the role of language in cognition? Is it merely an expression of thought and a vehicle for communication? Or does it determine the structure of thought? As for most deep philosophical questions, the answer is "Yes and no!" Yes, the basic structure in thought is grounded in the evolved structure of the sensory-motor system. No, there is more to the story. The structure of mental models, perhaps even of aspectual and image schemas, is shaped by experience with tools, linguistic as well as physical. In the following sections we consider evidence for this in physics and mathematics.

VIII. Concepts of Force in science and common sense

From the beginning, Modeling Theory was developed with an eye to improving instruction in science and mathematics, so we look to that domain for validation of the theory. In section II, I reported the stunning success of Malcolm Wells' initial experiment with Modeling Theory and its subsequent flowering in the Modeling Instruction Project. My purpose in this section is, first to describe what Modeling Theory initially contributed to that success, and second to propose new explanations based on the current version of the theory. This opens up many opportunities for further research.

School physics has a reputation for being impossibly difficult. The rap is that few have the talent to understand it. However, PER has arrived at a different explanation by investigating *common sense* (CS) concepts of force and motion in comparison to the Newtonian concepts of physics. The following conclusions are now widely accepted:

- CS concepts **dominate** student thinking in introductory physics!
- Conventional instruction is almost totally **ineffective** in altering them!
- This result is **independent** of the instructor's academic qualifications, teaching experience, and (unless informed by PER) mode of teaching!

Definitive quantitative support for these claims was made possible by development of the ***Force Concept Inventory*** (FCI). The initial results [3, 5] have been repeatedly replicated (throughout the U.S. and elsewhere), so the conclusions are universal, and only the ill-informed are skeptical.

The implications for conventional instruction could hardly be more serious! Student thinking is far from Newtonian when they begin physics, and it has hardly changed (<15%) when they finish the first course. Consequently, students systematically misinterpret almost everything they read, hear and see throughout the course. Evidence for this catastrophe has always been there for teachers to see, but they lacked the conceptual framework to recognize it.

Witness the common student complaint: "I understand the theory, I just can't work the problems!" In my early years of teaching I dismissed such claims as unfounded, because ability to work problems was regarded as the definitive test of understanding. Now I see that the student was right. He did understand the theory — but it was the wrong theory! His theory wrapped up his CS concepts in Newtonian words; he had learned *jargon* instead of Newtonian concepts.

Since students are oblivious to the underlying conceptual mismatch, they cannot process their own mistakes in problem solving. Consequently, they resort to rote learning and depend on the teacher for answers. A sure sign of this state of affairs in a physics classroom is student clamoring for the teacher to demonstrate solving more and more problems. They confuse memorizing problem solutions with learning how to solve problems. This works to a degree, but repeated failure leads to frustration and humiliation, self-doubt and ultimately student turn-off!

Happily, this is not the end of the story. Figure 10 summarizes data from a nationwide sample of 7500 high school physics students involved in the *Modeling Instruction Project* during 1995–98. The mean FCI pretest score is about 26%, slightly above the random guessing level of 20%, and well below the 60% score which, for empirical reasons, can be regarded as a *threshold* in the understanding of Newtonian mechanics.

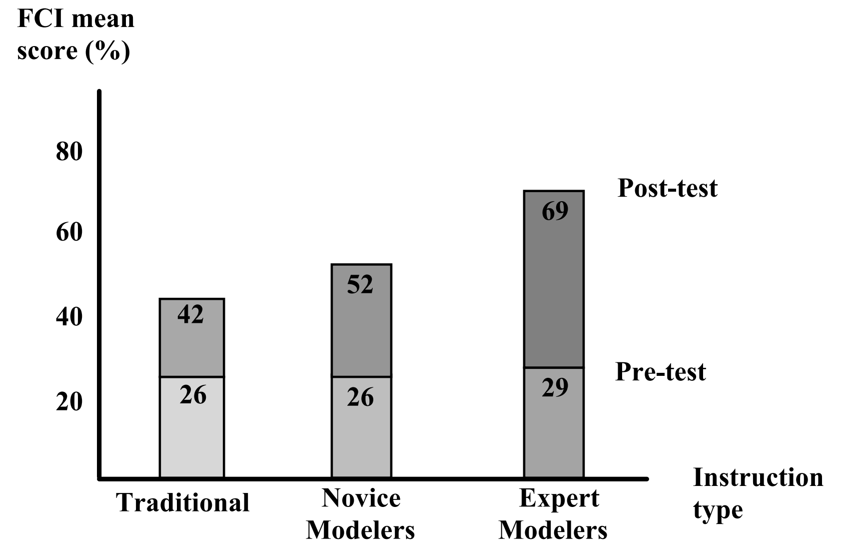
Figure 10 shows that traditional high school instruction (lecture, demonstration, and standard laboratory activities) has little impact on student beliefs, with an average FCI posttest score of 42%, still well below the Newtonian threshold. This is data from the classes of teachers before participating in the Modeling Instruction Project.

Participating teachers attend an intensive 3-week Modeling Workshop that immerses them in modeling pedagogy and acquaints them with curriculum materials designed expressly to support it. Almost every teacher enthusiastically adopts the approach and begins teaching with it immediately. After their first year of teaching posttest scores for students of these *novice modelers* are about 10% higher, as shown in Fig. 10 for 3394 students of 66 teachers. Students of expert modelers do much better.

For 11 teachers identified as expert modelers after two years in the Project, posttest scores of their 647 students averaged 69%. *Their average gain is more than two standard deviations higher than the gain under traditional instruction.* It is comparable to the gain achieved by the first expert modeler Malcolm Wells.

The **29%/69% pretest/posttest means for the expert modelers** should be compared with the **52%/63% means for calculus-based physics at a major university** [5]. We now have many examples of modelers who consistently achieve posttest means from 80-90%. On the other hand, even initially under-prepared teachers eventually achieve substantial gains, comparable to gains for well-prepared teachers after two years in the project.

FCI scores are vastly more informative than scores for an ordinary test. To see why, one needs to examine the structure of the test and the significance of the questions. The questions are based on a detailed taxonomy of *common* *sense* (CS) *concepts of force and motion* derived from research. The taxonomy is structured by a systematic analysis of the Newtonian force concept into six fundamental conceptual dimensions. Each question requires a forced choice between a Newtonian concept and CS alternatives for best explanation in a common physical situation, and the set of questions systematically probes all dimensions of the force concept. Questions are designed to be meaningful to readers without formal training in physics.



FCI mean scores under different instruction types

To a physicist the correct choice for each question is so obvious that the whole test looks trivial. On the other hand, virtually all CS concepts about force and motion are incompatible with Newtonian theory. Consequently, every missed question has high information content. Each miss is a sure indicator of non-Newtonian thinking, as any skeptical teacher can verify by interviewing the student who missed it.

Considering the FCI's comprehensive coverage of crucial concepts, the abysmal FCI scores for traditional instruction imply *catastrophic failure to penetrate student thinking!* Most high school students and half the university students do not even reach the Newtonian threshold of 60%. Below that threshold students have not learned enough about Newtonian concepts to use them reliably in reasoning. No wonder they do so poorly on problem solving.

Why is traditional instruction so ineffective? Research has made the answer clear. To cope with ordinary experience each of us has developed a loosely organized system of intuitions about how the world works. That provides intuitive grounding for CS beliefs about force and motion, which are embedded in natural language and studied in linguistics and PER. Research shows that CS beliefs are **universal** in the sense that they are much the same for everyone, though there is some variation among individuals and cultures. They are also very **robust** and expressed with confidence as obvious truths about experience.

Paradoxically, physicists regard most CS beliefs about force and motion as obviously false. From the viewpoint of Newtonian theory they are simply *misconceptions* about the way the world truly is! However, it is more accurate, as well as more respectful, to regard them as *alternative hypotheses*. Indeed, in preNewtonian times the primary CS "misconceptions" were clearly articulated and forcefully defended by great intellectuals — Aristotle, Jean Buridan,

Galileo, and even Newton himself (before writing the *Principia*) [4]. Here we see another side of the paradox:
To most physicists today Newtonian physics describes obvious structure in perceptible experience, in stark contrast to the subtle quantum view of the world. I have yet to meet a single physicist who recollects ever holding pre-scientific CS beliefs, though occasionally one recalls a sudden *aha!* insight into Newton's Laws. This *collective retrograde amnesia* testifies to an important fact about memory and cognition: recollections are reconstructed to fit current cognitive structures. Thus, physicists cannot recall earlier CS thinking because it is filtered by current Newtonian concepts.

In conclusion, the crux of the problem with traditional instruction is that it does not even recognize CS beliefs as legitimate, let alone address them with argument and evidence. In contrast, *Modeling Instruction* is deliberately designed to address this problem with

- *Modeling activities* that systematically engage students in developing models and providing their own explanations for basic physical phenomena,
- *Modeling discourse* (centered on visual representations of the models) to engage students in articulating their explanations and comparing them with Newtonian concepts,
- *Modeling concepts and tools* (such as graphs, diagrams and equations) to help students simplify and clarify their models and explanations.

Instructors are equipped with a taxonomy of CS concepts to help recognize opportunities to elicit the concepts from students for comparison with Newtonian alternatives and confrontation with empirical evidence. Instructors know that students must recognize and resolve discrepancies by themselves. Telling them answers does not work.

From years of experimenting with modeling discourse (especially in the classroom of Malcolm Wells) we have learned to focus on the *three CS concepts* listed in Box 5. When these concepts are adequately addressed, other misconceptions in our extensive taxonomy [5] tend to fall away automatically. Their robustness is indicated by the posttest discrepancies (Box 5) from FCI data on more than a thousand university students. After completing a first course in calculus-based physics, the fraction of students choosing CS alternatives over Newton's First, Second and Third Laws was 60%, 40% and 90% respectively. Of course, Newton's Laws are not

Box 5 Contrasting Force Concepts

Newtonian	vs.	Common Sense	Posttest Discrepancy
• First Law	⇔	"Motion requires force" (Impetus Principle)	~ 60%
• Second Law	⇔	"Force is action" (No Passive forces)	~ 40%
• Third Law	⇔	"Force is war" (Dominance Principle)	~ 90%

named as such in the FCI. 80% of the students had already taken high school physics and could state Newton's Laws as slogans before beginning university physics.

After the Modeling Instruction Project was up and running, I learned about Lakoff's work on metaphors and its relevance for understanding CS force and motion concepts. I presented the ideas to teachers in Modeling Workshops but have no evidence that this improved the pedagogy, which was already well developed. I suppose that much of the new insight was overlooked, because it was not nailed down in print, so let me record some of it here as analysis of the three *primary CS concepts* in Box 5.

The Impetus Principle employs the Object-As-Container metaphor, where the container is filled with impetus that makes it move. After a while the impetus is used up and the motion stops. Of course, students don't know the term *impetus* (which was coined in the middle ages); they often use the term *energy* instead. Naïve students don't discriminate between energy and force. Like Newton himself before the *Principia,* they have to be convinced that "free particle motion in a straight line" is a natural state that doesn't require a motive force (or energy) to sustain it. This does not require discarding the impetus intuition (which is permanently grounded in the sensory-motor system in any case) but realigning the intuition with physics concepts of inertia and momentum.

The **CS prototype for force is human action** on an object. Consequently, students don't recognize constraints on motion like walls and floors as due to contact forces. "They just get in the way." Teachers try to activate student intuition by emphasizing that "force is a push or a pull," without realizing that unqualified application of this metaphor excludes passive forces. Besides, no textbooks explicitly note that **universality of force** is an implicit assumption in Newtonian theory, which requires that motion is influenced only by forces. To arrive at force universality on their own, students need to develop intuition to recognize forces in any instance of physical contact. As an instructional strategy to achieve that end, Clement and Camp [28] engage students in constructing a series of "bridging analogies" to link, for example, the unproblematic case of a person pressing on a spring to the problematic case of a book resting on a table. I recommend modifying their approach to include a common vector representation of normal force in each case to codify symbolic equivalence (as in Fig. 4).

In situations involving Newton's Third Law, the slogan "for every action there is an equal and opposite reaction" evokes a misplaced analogy with a struggle between "opposing forces," from which it follows that one must be the winner, "overcoming" the other, in contradiction to the Third Law. The difficulty that students have in resolving this paradox is reflected in the fact that FCI questions on the Third Law are typically the last to be mastered. DiSessa [29] gives a perceptive analysis of Third Law difficulties and measures to address them.

Such insights into student thinking as just described are insufficient for promoting a transition to Newtonian thinking in the classroom. The literature is replete with attempts to address specific misconceptions with partial success at best. So what accounts for the singular success of Modeling Instruction as measured by the FCI (Fig. 10)? As for any expert performance, detailed planning and preparation is essential for superior classroom instruction. (The intensive Modeling Workshops help teachers with that.) However, Modeling Instruction is unique in its strategic design.

Rather than address student misconceptions directly, Modeling Instruction creates an environment of activities and discourse to stimulate reflective thinking about physical phenomena that are likely to evoke those misconceptions. The environment is structured by an emphasis on models and modeling with multiple representations (maps, graphs, diagrams, equations). This provides students with conceptual tools to sharpen their thinking and gives them access to Newtonian concepts. In this environment students are able to adjust their thinking to resolve discrepancies within the Newtonian system, which gradually becomes their own. Rather than learning Newtonian concepts piecemeal, they learn them as part of a coherent Newtonian system. Construction of a Newtonian model requires coordinated use of all the Newtonian concepts, and only this reveals the coherence of the Newtonian system. That coherence is not at all obvious from the standard statement of Newton's Laws. I believe that learning Newtonian concepts as a coherent system best accounts for high FCI scores. Logically this is only a sufficient condition for a high score, but I estimate that a high score from piecemeal understanding of Newtonian physics is improbably low. Thus, it is best to *interpret overall FCI score as a measure of coherence in understanding Newtonian physics.*

One other important point deserves mention here. As we have noted, Modeling Theory informed by empirical evidence from cognitive science holds that mental models are always constructed within a semantic frame. Accordingly, I suppose that physical situations (regardless of how they are presented) activate a Newtonian semantic frame in the mental spaces of physicists. And I submit that physics instruction is not truly successful until the same is true for students. It is well known that students tend to leave the science they have learned in the classroom and revert to CS thinking in every day affairs. Perhaps recognizing this as a problem of semantic framing can lead to a better result.

As I have described it, Modeling Instruction does not depend on detailed understanding of how students think. Indeed, I have tried to steer it clear of doubtful assumptions about cognition that might interfere with learning. However, I now believe that advances in the Modeling Theory of cognition described in Section VII are sufficient to serve as a reliable guide for research to further improve instruction by incorporating details about cognition. Let me sketch the prospects with specific reference to force and motion concepts.

The intertwined concepts of *force and causation* have been studied extensively in cognitive linguistics. Lakoff and Johnson [19] show that the great variety of causal concepts fall naturally into a radial category ("kinds of causation") structured by a system of metaphorical projections. The central prototype in this category is given by the Force-as-Human-Action metaphor, in agreement with our analysis above. Their analysis provides an organizational framework for the whole body of linguistic research on causation. That research provides valuable insight into CS concepts of force and motion that deserves careful study. However, limited as it is to study of natural languages, linguistic research does not discover the profound difference in the force concept of physicists. For that we need to turn to PER, where the deepest and most thorough research is by Andy diSessa [29].

In much the same way that linguists have amassed evidence for the existence of prototypes and image schemas, diSessa has used interview techniques to isolate and characterize conceptual primitives employed by students in causal reasoning. He has identified a family of irreducible "knowledge structures" that he calls *phenomenological primitives* or **p-prims**. Since diSessa's definitive monograph on p-prims in 1993, converging evidence from cognitive linguistics has made it increasingly clear that his p-prims are of the same ilk as the image and aspectual schemas discussed in the preceding section. Accordingly, I aim to integrate them under the umbrella of Modeling Theory.

Let us begin with the most important example, which diSessa calls Ohm's p-prim. As he explains,

> **Ohm's p-prim** comprises "an <u>agent</u> that is the locus of an <u>impetus</u> that acts against a <u>resistance</u> to produce a <u>result</u>."

Evidently this intuitive structure is abstracted from experience pushing objects. It is an important elaboration of the central Force-as-Action metaphor mentioned above — *Very important!* — Because this structure is fundamental to qualitative reasoning. The logic of Ohm's p-prim is the *qualitative proportion*:

more effort $\Rightarrow$ more result,

and the *inverse proportion*:

more resistance $\Rightarrow$ less result.

This reasoning structure is evoked for explanatory purposes in circumstances determined by experience.

DiSessa identifies a number of other p-prims and catalogs them into a cluster that corresponds closely to the taxonomy of CS force and motion concepts used to construct the FCI. His monograph should be consulted for many details and insights that need not be repeated here. Instead, I comment on general aspects of his analysis.

In accord with Lakoff and Johnson, diSessa holds that causal cognition is grounded in a loosely organized system of many simple schemas derived from sensory-motor experience. P-prims provide the grounding for our intuitive sense of (causal) mechanism. They are the CS equivalent of physical laws, used to explain but not explainable. To naïve subjects, "that's the way things are."

As to be expected from their presumed origin in experience, p-prims are cued directly by situations without reliance on language. DiSessa asserts that p-prims are inarticulate, in the sense that they are not strongly coupled to language. Here there is need for further research on subtle coupling with language that diSessa has not noticed. For example, Lakoff notes that the preposition *on* activates and profiles schemas for the concepts of *contact* and *support*, which surely should be counted among the p-prims.

As disclosed in Ohm's p-prim, the concept of (causal) **agency** entails a basic

Causal syntax: agent → (kind of action) → on patient → result.

DiSessa notes that this provides an interpretative framework for $\mathbf{F} = m\mathbf{a}$, and he recommends exploiting it in teaching mechanics. However he does not recognize it as a basic aspectual schema for verb structure, which has been studied at length in cognitive grammar [22]. Aspectual concepts are generally about event structure, where events are changes of state and causes (or causal agents) induce events. Causes cannot be separated from events. Here is more opportunity for research.

Under physics instruction, diSessa says that p-prims are refined but not replaced, that they are gradually tuned to expertise in physics. Considering the role of metaphor and analogy in this process, it might be better to say that p-prims are realigned. There are many other issues to investigate in this domain. Broadly speaking, I believe that we now have sufficient theoretical resources to guide research on instructional designs that target student p-prims more directly to retune and integrate them into schemas for more expert-like concepts. I propose that we design *idealized expert prototypes* for force and motion concepts to serve as targets for instruction. This would involve a more targeted role for diagrams to incorporate figural schemas into the prototypes.

The call to design expert prototypes embroils us in many deep questions about physics and epistemology. For example, do forces really exist outside our mental models? We have seen that Modeling Theory tells us that the answer depends on our choice of theoretical primitives and measurement conventions. Indeed, if momentum is a primitive, then Newton's Second Law is reduced to a definition of *force as momentum flux* and the Third Law expresses momentum conservation. The physical intuition engaged when mechanics is reformulated in terms of momentum and momentum flux has been investigated by diSessa among others, but few physicists have noted that fundamental epistemological issues are involved. Not the least of these issues is the transition from classical to quantum mechanics, where momentum is king and force is reduced to a figure of speech.

A related epistemological question: Is causal knowledge domain-specific? Causal claims are supported by causal inference from models based on acquired domain-specific knowledge. But to what degree does inference in different domains engage common intuitive mechanisms? Perhaps the difference across domains is due more to structure of the models rather than the reasoning. Perhaps we should follow Lakoff's lead to develop *force and interaction* as a radial category for a progression of interaction concepts ranging from particles to fields.

I am often asked how the FCI might be emulated to assess student understanding in domains outside of mechanics, such as electrodynamics, thermodynamics, quantum mechanics and even mathematics. Indeed, many have tried to do it themselves, but the result has invariably been something like an ordinary subject matter test. The reason for failure is insufficient attention to cognitive facts and theory that went into FCI design, which I now hope are more fully elucidated by Modeling Theory. The primary mistake is to think that the FCI is basically about detecting misconceptions in mechanics. Rather, as we have seen, it is about comparing CS causal concepts to Newtonian concepts. The p-prims and image schemas underlying the CS concepts are not peculiar to mechanics, they are basic cognitive structures for reasoning in any domain. Therefore, the primary problem is to investigate how these structures are adapted to other domains. Then we can see whether reasoning in those domains requires other p-prims that have been overlooked. Finally, we can investigate whether and how new p-prims are created for advanced reasoning in science and mathematics. That brings us to the next section, where we discuss the development of conceptual tools to enhance scientific thinking.

IX. Tools to think with

The evolution of science is driven by invention and use of tools of increasing sophistication and power! The tools are of two kinds: instruments for detecting patterns in the material world, and symbolic systems to represent those patterns for contemplation. As outlined in Fig. 11, we can distinguish three major stages in tool development.

In the perceptual domain, pattern detection began with direct observation using human sensory apparatus. Then the perceptual range was extended by scientific instruments such as telescopes and microscopes. Finally, human sensory detectors are replaced by more sensitive detection instruments, and the data are processed by computers with no role for humans except to interpret the final results; even there the results may be fed to a robot to take action with no human participation at all.

Tool development in the cognitive domain began with the natural languages in spoken and then written form. Considering their *ad hoc* evolution, the coherence, flexibility and subtlety of the natural languages is truly astounding. More deliberate and systematic development of symbolic tools came with the emergence of science and mathematics. The next stage of enhancing human cognitive powers with computer tools is just beginning. My purpose in this section is to discuss what Modeling Theory can tell us about the intuitive foundations of mathematics to serve as a guide for research on design of better instruction and better mathematical tools for modeling in science and engineering.

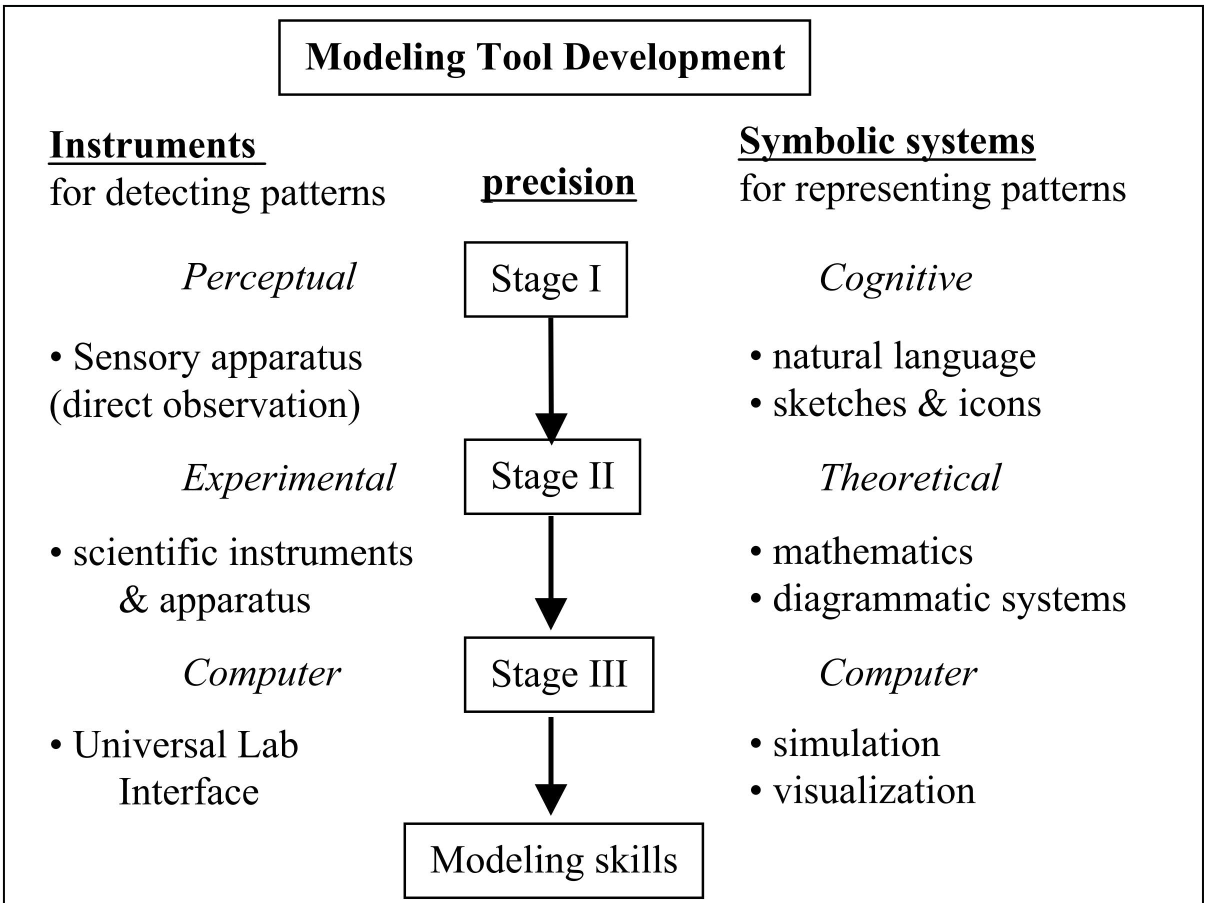


Fig. 11

While science is a search for structure, mathematics is the science of structure. Every science develops specialized modeling tools to represent the structure it investigates. Witness the rich system of diagrams that chemists have developed to characterize atomic and molecular structure. Ultimately, though, these diagrams provide grist for mathematical models of greater explanatory power. What accounts for the ubiquitous applicability of mathematics to science?

I have long wondered how mathematical thinking relates to theoretical physics. According to Modeling Theory, theoretical physics is about designing and analyzing conceptual models that represent structure in the material world. For the most part these models are mathematical models, so the cognitive activity is called mathematical modeling. But how does mathematical thinking differ from the mathematical modeling in physics? Can it be essentially the same when there are no physical referents for the mathematical structures? *I am now convinced that the answer is yes*! The light went on when I learned about cognitive semantics and realized that **the referents for cognition in both mathematics and physics are mental models!** Lakoff and Núñez [20] argue forcefully for the same conclusion, but I want to put my own twist on it.

I contend that the basic difference between mathematics and physics is how they relate their mental models to the external world. Physicists aim to match their mental models to structure in the material world. I call the ability to make such matches **physical intuition.** Note that mathematics is not necessarily involved in this. In contrast, mathematicians aim to match their mental models to structure in symbolic systems. I call the ability to make such matches **mathematical intuition**. To be sure, physicists also relate their mental models to mathematical structures, but for the most part they take the mathematics as given. When they do venture to modify or extend the mathematical structures they function as mathematicians. Indeed, that is not uncommon; a vast portion of mathematics was created by theoretical physicists.

According to Modeling Theory, mathematicians work with intuitive structures (grounded in sensory-motor experience) that every normal person has. They proceed to encode these structures in symbolic systems and elaborate them using the intuitive inferential structures of p-prims and image schemas. I submit that mathematical thinking involves a feedback loop generating external symbolic structures that stimulate modeling in mental spaces to generate more symbolic structure. Though some mathematical thinking can be done with internal representations of the symbols, external representation is essential for communication and consensus building [30]. For this reason, I believe that the invention of written language was an essential prerequisite to the creation of mathematics.

Let's consider an example of intuitive grounding for mathematical structures. Lakoff and Núñez [20] give many others, including four grounding metaphors for arithmetic. Note that the intuitive causal syntax discussed in the previous section can be construed (by metaphorical projection at least) as

Operator syntax: agent → (kind of action) → on patient → result,

where the action is on symbols (instead of material objects) to produce other symbols. Surely this provides an intuitive base for the mathematical concept of function (though it may not be the only one). Exploration of mental models reveals various kinds of structure that can be encoded and organized into symbolic systems such as Set theory, Geometry, Topology, Algebra and Group theory. Note that the number of distinct types of mathematical structure is limited, which presumably reflects constraints on their grounding in the sensory-motor system. Of course, to confirm this point of view thorough research is needed to detail the intuitive base for each type of mathematical structure. Lakoff and Núñez [20] have already made a good start.

The upshot is that cognitive processes in theoretical physics and mathematics are fundamentally the same, centered on construction and analysis of conceptual models. Semantics plays a far more significant role in mathematical thinking (and human reasoning in general) than commonly recognized — it is the *cognitive semantics* of mental models, mostly residing in the cognitive unconscious, but often manifested in pattern recognition and construction skills [31]. Mathematical intuition (like physical intuition) is a repertoire of mental structures (schemas) for making and manipulating mental models! This goes a long way toward answering the question: What does it mean to *understand* a scientific concept?

I am not alone in my opinion on the intimate relation between physics and mathematics. Here is a brief extract from a long diatribe *On Teaching Mathematics* by the distinguished Russian mathematician V. I. Arnold [32]:

> "Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap. . . . In the middle of the 20th century it was attempted to divide physics and mathematics. The consequences turned out to be catastrophic. Whole generations of mathematicians grew up without knowing half of their science and, of course in total ignorance of other sciences."

Arnold is deliberately provocative but not flippant. He raises a very important educational issue that deserves mention quite apart from the deep connection to cognitive science that most concerns us here.

There is abundant evidence to support Arnold's claim. For example, up until World War II physics was a required minor for mathematics majors in US universities. Since it was dropped, the mathematics curriculum has become increasingly irrelevant to physics majors, and physics departments provide most of the mathematics their students need. At the same time, mathematicians have contributed less and less to physics, with some exceptions like the Russian tradition that Arnold comes from, which has sustained a connection to physics. But the most serious consequence of the divorce of mathematics from physics is the fact that, in the U.S. at least, most high school math teachers have little insight into relations of math they teach to science in general and physics in particular. Here is a bit of data to support my contention: We administered the FCI to a cohort of some 20 experienced high school math teachers. The profile of scores was the same as the pitiful profile for traditional instruction in Fig. 10, with the highest score at the Newtonian threshold of 60%. Half the teachers missed basic questions about relating data on motion to concepts of velocity and acceleration. This chasm between math and science, now fully ensconced in the teachers, may be the single most serious barrier to significant secondary science education reform.

To document deficiencies in math education, many have called for a *Math Concept Inventory* (MCI) analogous to the FCI. I have resisted that call for lack of adequate theory and data on intuitive foundations for mathematical thinking. There is lots of educational research on conceptual learning in mathematics, but most of it suffers from outdated cognitive theory. Modeling Theory offers a new approach that can profit immediately from what has been learned about cognitive mechanisms in physics. We need to identify **"m-prims"** that are mathematical analogs of the p-prims discussed in the preceding section. I suspect that underlying intuitive mechanisms are the same for m-prims and p-prims, but their connections to experience must be different to account for the difference between mathematical and physical intuition noted above. I recommend coordinated research on m-prims and p-prims aiming for a comprehensive Modeling Theory of cognition in science and mathematics.

I have barely set the stage for application of Modeling Theory for my favorite enterprise, namely, the design of modeling tools for learning and doing science, engineering and mathematics [10]. I have previously described the influence of my Geometric Algebra research on development of Modeling Theory [13]. Now I believe that Modeling Theory has matured to the point where it can contribute, along with Geometric Algebra, to the design of more powerful modeling tools, especially tools embedded in computer software. But that is a task for tomorrow!

X. Conclusion

Central thesis: Cognition in science, mathematics, and everyday life is basically about making and manipulating *mental models*.

- The human cognitive capacity for creating, manipulating and remembering ***mental models*** has evolved to facilitate coping with the environment, so it is central to "common sense" thinking and communication by humans.
- Human culture has expanded and augmented this capacity by creating ***semiotic systems***: representational systems of signs (symbols, diagrams, tokens, icons, etc.), most notably spoken and written language.
- Science and mathematics has further extended the use of symbolic systems deliberately and self-consciously, but the *cognitive mechanisms involved are essentially the same as for common sense.*

Scientific modeling is a "deliberate and self-conscious extension of the evolved cognitive capabilities for "mapping" the environment." (Giere)

Science is a refinement of common sense! differing in respect to:

Objectivity – based on explicit rules & conventions for observer-independent inferences

Precision – in measurement
– in description and analysis

Formalization – for mathematical modeling and analysis of complex systems

Systematicity – coherent, consistent & maximally integrated bodies of knowledge

Reliability – critically tested & reproducible results

Skepticism – about unsubstantiated claims

Knowledge and Wonder – so say Weisskopf & Sagan

Social structure and norms – Ziman

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